This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

NYPL RESEARCH LIBRARIES
3 3433 06641862 9

STRENGTH

OF

WROUGHT-IRON BRIDGE MEMBERS.

PART I.—GENERAL THEORY OF BEAMS.

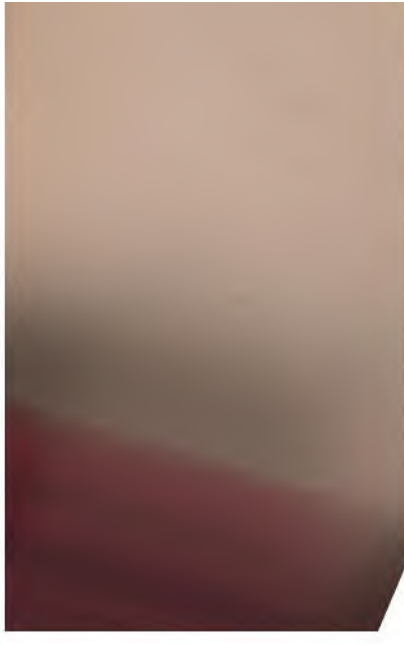
PART II.—PRACTICAL FORMULAS FOR BEAMS, STRUTS, COLUMNS AND SEMI-COLUMNS.—EXTENDED COMPARISON OF VARIOUS FORMULAS WITH EXPERIMENT.

By S. W. ROBINSON, C.E.,

Prof. Mech. Eng., Ohio State University, and of the Ohio Board of Inspectors of Railroads and Bridges.

REPRINTED FROM VAN NOSTRAND'S MAGAZINE.

NEW YORK:
D. VAN NOSTRAND, PUBLISHER,
23 MURRAY AND 27 WARREN STREET.
1882.



PREFACE.

THE examination of existing bridges for strength and trustworthiness has occasioned the use of formulas not accessible to the writer in published form. In procuring the formulas by direct solution, the amount of labor entailed has been great. The avoidance of a repetition of this labor, on the part of others who may require these formulas, is believed to be sufficient reason for now publishing the results obtained.

The formulas referred to are such as take account of longitudinal as well as simultaneously acting transverse loads, while the beam is itself held in various ways.

to vanish altogeth
for ordinary beams
semi-columns, &c., s

Thus, expression
of rational formul
been obtained—for
transcending the o
mulas in coincidenc
experimental result
known the compo
furnished a true ke
loads.

Also the effect of
bearings has been

H. Sabine, Commissioner of Railroads and Telegraphs for Ohio, in which service results were sought, in critical cases, that were worthy of the utmost possible confidence. S. W. R.

OHIO STATE UNIVERSITY,
April 24, 1882.

Strength of Wrought-Iron Bridge Members.

I.—General Theory of Beams.

Examples of beams coming under a more general theory than that of mere transverse loading are found in bridges; the same, for complete solution, requiring unusual formulas.

To make clear the nature of these formulas, the conditions to which certain bridge members are subject may be referred to. Thus, in some instances, the chords of truss bridges are required to carry the floor beams, one, two or more of them, to each panel. Now, when a train of cars comes upon a bridge, the floor-beam loads rest down upon the chord members and deflect them into downward bowing curves, causing "transverse strains," or "bending moments." Simultaneously, the load upon the bridge causes endlong strains in the same chord members; tension for the lower, and compression for the upper

...ent. Thus, if individual membe...
...ure, such as an "...
...ted as a beam subje...
...ompressed, bending and stretch...
...ding and compression, as the ...
...ave ...

The term may be alternating the ...
...ximum stress in the piece ...
...ional, as it commonly is, though the b...
...ly ... stress,' and ...
...ompletely ... the end ...
...stress, and add the results. It is evi...
...however, from a static consideration ...
...is the bending load would separ...
...give a certain curve to the beam. ...
...tension would partially straighten ...
...curve and diminish the bending mom...
...a compressive buckling st...
...ace of the beam would b...
...the bending moment ...
...the usual ...

The conditions of the piece just referred to as regards loading and supports are stated in general terms, but it is evident that they include all such special conditions as concentrated loads; distributed loads; one end fixed and the other free; one end fixed and the other supported; both ends supported; both ends fixed, &c.

TRUE FLEXURAL MOMENT AND ITS RESISTANCE IN MIXED STRAINS.

Before proceeding to special cases, it is necessary to consider the pseudo-philosophy of eccentric neutral axis in beams subject to both endlong and cross forces.

For the sake of fixing the ideas, let Fig. 1 represent a beam fixed at the end, B, and free at the end, A. Let the cross force, P, act at right angles to the beam, and the endlong force, T, in the direction of the length, both being applied at the free end of the beam. The bending will be as shown, the fibers on the upper side of the beam being stretched, and those on the lower side being compressed, if T is not too great.

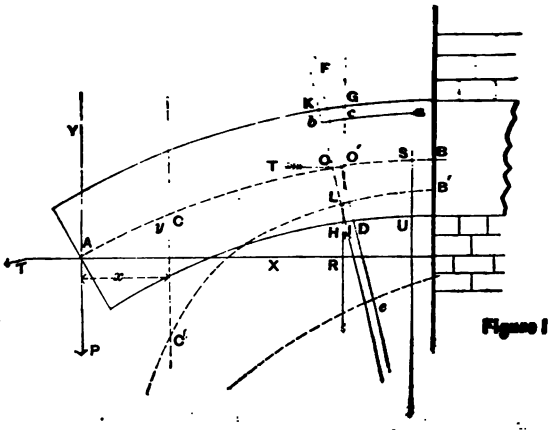
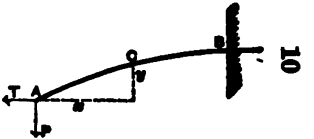


Figure I



Now, undoubtedly one effect of T is to elongate the whole beam and every part of it, throwing the neutral axis for all the stresses, downward from the line ACB to some line C′B′, or C″B″ This displacement is proportional to $\frac{T}{Px}$ for the section at the distance x, from the free end of the beam. If $T=o$, the displacement$=o$, anywhere. If $P=o$, the displacement is ∞, anywhere. If $x=o$, the displacement $= \infty$. If $x=l$, and P very small, the displacement is very great. But for T and P moderate, the displacement is moderate. For T and P constant and x variable, the departure of the line of the neutral axis, C′B′ from CB, increases as x decreases, and, geometrically speaking, it is asymptotic to the horizontal through B, and to the vertical, through A. Though this line is a real *neutral* axis, as regards stress, it can easily be shown that it is not the most convenient line in which to place the origin of moments, for the well known differential equation of the elastic curve, viz.:

This equation, as it st
tion of moments of for
member being the sum
of the forces acting on
the 1st member is the
ments of the internal fo
resisting bending, or, r
"moment of resistance
The origin of moments,
for this equation, is tak
tain cross section of th
stance, DF, and then th
forces are to be expres
ular origin of momen
is then integrated, res
tion of the elastic cu

ered, no one doubts; but *where* it should be taken is seriously questioned. Some writers definitely locate it, by saying that the moment of inertia, I, is here variable, because the neutral axis, which is supposed coincident with the axis for I, is not parallel to the beam. According to this, if C'B' be the neutral axis, as above described, the origin of moments must be taken at L, in applying the above general equation (1) to Fig. 1. The moment of inertia, I, of the section, DF, is then to be expressed for the moment of inertia axis at L, coinciding with the origin of moments. Now, to discuss this conclusion, draw G H, DF, and JK, as shown. Let the portion, SO, of the beam be supposed to stretch, displacing DF to JK, this stretch being due to T. Again, let the bending moments displace GH, through an angle to FD, causing a stretch in the upper fibers and a compression in the lower fibers of the beam, the triangles, O'FG and O'HD, representing the two actions respectively. Respecting a fiber,

for the sake of an arg
be granted that the origin
for equation (1) be taken
neutral axis, C'B', at L. Th
where the cross section, JK
intersects the real neutral axis
L is the point in the section,
there is actually no longitudi
The figure indicates this, since
of the beam, GF stands for s
to cross strain; FK, the stretc
long strain, their sum, GK,
total stretch or stress. This st
and less in going down from
until it vanishes at L, where th
lines, GL and KL, for the stre
intersect on the neutral axis, C

Now, in applying

where the moment of inertia, I, of the first member, is to be so expressed that its axis coincides in position with the origin of moments, L. Let the beam be regarded as uniform in cross section from end to end, with the line, ACOB traversing the center of gravity of any cross section. Let I_1 represent the moment of inertia of the cross section considered, for an axis located at the center of gravity of that section. For KJ, this axis is at O, the point O being the "principal axis" or "center of inertia." Now, a well-known principle in mechanics, by which I may be obtained from I_1, gives

$$I = I_1 + \text{section JK}.\overline{LO}^2 \quad . \quad (3).$$

Hence the first member of equation (2) is

$$\frac{\varepsilon I}{\rho} = \frac{\varepsilon I_1}{\rho} + \frac{\varepsilon}{\rho} \text{ section JK}.\overline{LO}^2$$

$$= \frac{\varepsilon I_1}{\rho} + \frac{\varepsilon}{\rho} K.\overline{LO}^2 \quad . \quad . \quad (4).$$

where K is here taken for the cross section JK.

The denominator ρ is the radius of

S O′ and $d\,l$=O′ O, the section, K, being the same whole section, JK. In this, the dl is=O′ O=F K=D J.

Hence

$$T.SO' = \varepsilon K.O'O = \varepsilon K.\frac{LO}{\rho}.SO'$$

or

$$T = \varepsilon K\frac{LO}{\rho}$$

combining this with (4), we get

$$\frac{\varepsilon I}{\rho} = \frac{\varepsilon I_1}{\rho} + T.LO$$

Comparing with equation (2) we have

$$\frac{\varepsilon I}{\rho} = \frac{\varepsilon I_1}{\rho} + T.LO = P.AR - T.RL$$

or

$$\frac{\varepsilon I_1}{\rho} = P.AR - T.RL - T.LO$$

$$= P.AR - T\,(RL + LO)$$

or

$$\frac{\varepsilon I_1}{\rho} = \varepsilon I_1\frac{d^2y}{dx^2} = P.AR - T.RO. = M$$

$$= \Sigma Px - \Sigma Ty \quad . \quad . \quad (5).$$

This equation has for its first member the ordinary expression for the moment of elastic resistance of beams, subject

... considered, or at O, Fig. 1
econd member expresses simply th
ients of the applied forces for the
igin of moments, viz.: the point C
Hence it follows that in cases of
e of beams, where part of the force
plied in an endlong or longitudin
ction, the moments may unhesitati
expressed with reference to an o
moments located at the center of g
of the cross section conside
uation (5) makes this evident,
y for the moment of resistance
the entire number of the app

n the above transformation the radius of c
ρ has been all the time regarded as of the
. Fig. 1 shows that this is correct. For ir

forces. Indeed, a little consideration of M, in equation (5), shows that the above statement respecting the moment of the applied forces is true in general; that is to say, it is true for any number of forces, having any conceivable points of application, and having any conceivable lines of direction. For each force may be resolved into two components, one longitudinal and the other transverse such as T.RO and P.AR respectively and added, ΣTy being the algebraic sum of the moments of all the former, and ΣPx of the latter. It is to be understood that in the integrations, proper limits are to be chosen, so that no point of concentration of applied force be overstepped in this, any more than in ordinary problems.

These facts place all problems in the resistance of beams on the same basis, that is to say, equation (5) applies with equal truth and exactness for a beam subject to the action of any system of forces, however complex, as for the simple case of a single transverse force; I_1

ing always regarded as the principal
oment of inertia and constant for a
beam of uniform cross section.

This complete generalization of equa-
tions (1) and (5) for a constant value of I
is important, since the integrations are
not only facilitated, but we are enabled
to apply the resulting formulas for a
beam under any system of forces, with
the same degree of confidence as for the
most simple case.

To confirm the facts above stated by
discussion, let us refer to Fig. 1. First
suppose P applied, bending the bea
downward as shown. Then apply T
shown. This latter will evidently
tially straighten the curvature, and
the end of the beam. But it will fi
come to rest under some degree of c
ture, P and T both acting. The po
S O′ of the beam will stretch, D F
ing to J K. Next apply an equ
endlong force—T, at the
O, of the cross-sectio
will have no tendenc
he portion S O, b

exactly return J K to D F. In this the end of the beam will not be raised or lowered, the deflection being entirely due to the two forces acting at the end of the beam. The stress in the beam is now simply that represented by the triangles O′ G F and O′ D H. The moment of resistance is $\frac{\varepsilon I_1}{\rho}$ axis for I_1 being at O′, and the moment of P will be P.AR. But as to T we have a *couple* between T at the line AR and—T at O′ the arm of the couple being RO. Hence the moment of this couple is T.RO′. All these moments are seen to have the origin of moments at the point O, or O′ both being practically the same point. This re-establishes equation (5).

So far, the endlong force, when spoken of in connection with Fig. 1, has been treated as producing tension. But it is easily seen that similar reasoning follows for T, so taken as to compress the beam. Indeed, the algebraic sum, ΣTy, mentioned above, contemplated positive as well as negative forces.

Equation (5) enables us to find the equation of the elastic curve of the bent beam. Having the equation of the curve, we can readily find the deflection at any point. Also the moment of strain M, may be found as soon as the unknown constants of integration are determined.

TRUE MAXIMUM STRAIN IN ANY CROSS SECTION.

By aid of the moment M and the end-long force T, we may find the maximum strain produced in any cross section of the beam, as for instance in the section at O, Fig. 1. At the top of the beam, the strain is GK, and at the bottom it is JH; GK being the greater, it is the one sought.

The strain GK is the sum of two parts, GF and FK, the former being due to the moment M of bending, and the latter to the direct tension T. To find the total amount of this strain per square inch, we may apply the well known formula for the strength of columns, finding the part GF thus

$$\frac{t_1 I_1}{d_1} = M, \quad . \quad . \quad . \quad . \quad (6).$$

where M is given by equation (5), as the algebraic sum of the moments of all the applied forces, the center of gravity of the section considered being the origin of moments, I_1 the *principal* moment of inertia for the same section, as previously stated, d_1 the distance from the center of gravity of the section to that point where the strain is greatest, or to where rupture or failure begins, as the case may be, and t_1 is the strain due to the moment M. In Fig. 1 O′ is the center of gravity of the section DF considered, I_1 is the moment o inertia of that section for O′ the axis, and d_1 is O′F. As a formula for determining the moment of ultimate resistance of beams, t_1 is to be assumed equal the ultimate resistance or modulus of resistance to rupture. Equation (6) is the converse of this, where M is predetermined and t_1 found as a stress considerably below the ultimate value. In calculating the ultimate strength of

beams, formula (6) is known to give values wide of the truth. This is due to the fact that the law of resistance in the neighborhood of rupture deviates greatly from the law of perfect elasticity. But within the elastic limits, as will usually be the case with (6), the formula, as is well known, will be truthful. The strain FK Fig. 1, as above explained is due to T, and if the intensity be represented by t_2 and the sectional area DF by K, we will evidently have

$$t_2 K = T \quad . \quad . \quad . \quad . \quad (7)$$

Let $t_1 = GK$, Fig. 1, stand for the whole maximum strain per square inch in the section considered.

Then

$$t = t_1 + t_2 = \frac{M d_1}{I_1} + \frac{T}{K} \quad . \quad . \quad . \quad (8).$$

a perfectly general formula, free from all assumed approximations except in ance where ρ is placed equal the lifferential co-efficient. This, is done by all writers on beams. that the error in this

assumption is of no practical import whatever in beams as ordinarily used. It is zero where the bent beam is not inclined to the axis x, and for an inclina- of 10° it is less than 5 per cent.

It is to be observed that for rupturing strains d_1 is to be taken as the distance from the center of gravity of the section to the point where the rupture takes place, whether it be by tension or compression.

When the beam is rectangular in cross section with a breadth b and depth d, we have

$$d_1=\frac{d}{2},\ I_1=\frac{bd^3}{12},\ \text{and}\ K=bd,$$

and hence (8) becomes,

$$t=\frac{6M}{bd^2}+\frac{T}{bd} \quad . \quad . \quad . \quad (9).$$

Here d_1 disappears, as indeed it will in all cases where the cross section of the beam is symmetrical with respect to an axis to that section taken horizontal through O′ Fig. 1.

It is often desirable to so ass
point of application of T, as to p
beam under the least possible m
stress when the structure contain
the beam receives its maximum
For instance, suppose the lower
of a bridge carries the floor beam
eral to each panel. Then it is ev
that T may be applied at such poi
to make its moment, with respect to
section of beam in question, equal
opposite to the moment of P. T
makes M = 0, and consequently $t_1 =$
This is evidently the best conditi
since t_s and T are sole and mutual d
pendencies, and, at best, the minimu
strain can never fall below t_s. This
and also equation (8) agree in making
$t_1 = 0$ and hence, for minimum strains,

M = o, and $t_1 = o$.

rate further, take an "eye
lower chord of any bridge.
represent the loads due to

floor beams. The section at the middle of the length will receive the greatest moment of strain due to ABC, &c., while the tension T is uniform throughout. Now, that $t_1 = o$, the centers of the eyes must be raised a distance ab, shown at the middle section, so that for the maximum bridge load,

$$T.ab = \left\{ \begin{array}{l} \text{sum of the moments of AB and} \\ \text{P, with respect to } a \end{array} \right. \quad . \; . \; (10).$$

When this condition is exactly satisfied, the middle part of the beam near JK will be simply stretched and not sprung into curvature. But if the body of the bar be straight when free from strain, it will, when under the condition, (10) be sprung at points near the eyes. The problem then becomes complex because ab varies with the spring, and would require to be determined with due regard to the amount of flexure, as subsequently done in this article.

But it is possible to so curve the bar in the manufacture, that any part of it will be simply stretched when bearing its

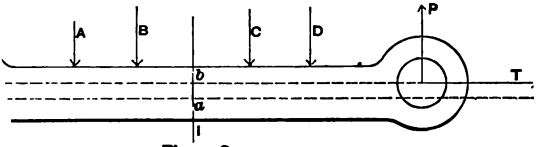


Figure 2.

a straight line from a, Fig. 2, each way, to the centers of the eyes, the distance ab being formed by (10). If there were two floor beams on the link, say A and D only, then the link should be straight between A and D, and also between these points and the centers of the eyes, the lines from point to point being constantly at mid-depth of the body of the bar. If the floor beams were so numerous as to be treated as constituting a uniformly distributed load, the curve of the center line of the body of the link would be a parabola running from center to center of eyes, with vertex at a, and with the principal axis vertical. Equation (10) would still determine ab, the 2d member being regarded as the sum of the movements of all forces except T. For this case the equation would take the form,

$$T.ab = P\frac{l}{2} - \frac{wl^2}{8} \text{ for the middle point,}$$

and (11).

$$Ty = Px - \frac{wx^2}{2} \text{ for any point,}$$

the origin of co-ordinates being at the middle of one eye, with l the length of the bar, and w the loading per unit length.

In cases where the cross section of the beam is non-symmetrical, as often the case in upper chords of bridges, the center of gravity of that section is to be found and then ab laid off, downward in the upper chord and upward in the lower chord.

When chord members are very long, the moment due to their own weight should be found and ab determined by (11).

GENERAL INTEGRALS FOR SOLVING CASES OF MIXED STRAINS.

In the designing of new structures the precautions pointed out above relative to maximum stresses in mixed strains may be provided for in the simplest way, viz: by computing separately and adding, although this may not always be preferable. this approximate method is not le, and in cases where existing

structures are to be examined, it
come necessary to make a carefu
mination of the moment M. T
this requirement the various cases
ing have been worked out.

Before taking up the special c
examination of the expressions
moments of applied forces, sho
the integrations all come und
general forms, and it will be see
decidedly preferable to perform t
integrations once for all than to in
for each special case.

The first general form is

$$\frac{d^2y}{dx^2} = Ay + Bx + Dx^2 + F$$

Differentiate twice, and

$$\frac{d^4y}{dx^4} = A\frac{d^2y}{dx^2} + 2D.$$

Now put

$$\frac{d^2y}{dx^2} = n,$$

and differentiate twice,

$$\frac{d^4y}{dx^4} = \frac{d^2n}{dx^2} = A\frac{d^2y}{dx^2} + 2D$$

and we get

$$\frac{dn^2}{dx^2} = An^2 +$$

or

$$dx = \frac{d}{\sqrt{A}\sqrt{n^2 + \frac{4I}{A}}}$$

whence, by integrati

$$x = \frac{1}{\sqrt{A}} \text{hyp. log} \frac{2D}{A}$$

Passing to expone ing in part,

$$C_1 e^{x\sqrt{A}} - \frac{2D}{A} - n =$$

By squaring both

solving for n we obtain

$$\left.\begin{aligned} n=&\frac{C_1}{2}e^{x\sqrt{A}}-\frac{C}{2C_1A}e^{-x\sqrt{A}} \\ &-\frac{2D}{A}+\frac{2D^2}{C_1A^2}e^{-x\sqrt{A}} \\ =&\frac{d^2y}{dx^2}=Ay+Bx+Dx^2+F \end{aligned}\right\} (13)$$

The second general form is

$$\frac{d^2y}{dx_2}=-Ay-Bx-Dx^2-F \quad . \quad . \quad (14).$$

Differentiate twice and

$$\frac{d^4y}{dx^4}=-A\frac{d^2y}{dx^2}-2D$$

Put

$$\frac{d^2y}{dx^2}=+n$$

and differentiate twice

$$\frac{d^4y}{dx^4}=\frac{d^2n}{dx^2}=-A\frac{d^2y}{dx^2}-2D$$

or

$$\frac{d^2n}{dx^2}=-An-2D$$

Multiply through by dn and integrate and

$$\frac{dn^2}{dx^2} = -An^2 - 4Dn + C'$$

or

$$dx = \frac{dn}{\sqrt{C' - 4Dn - An^2}}$$

But

$$\int \frac{dz}{\sqrt{a \pm bz - cz^2}} = \frac{1}{\sqrt{c}} \sin^{-1} \frac{2c}{\sqrt{4}}$$

where $a = C'$, $b = 4D$, $c = A$, $z =$

Hence

$$x = \frac{1}{\sqrt{A}} \sin^{-1} \frac{2An + 4I}{\sqrt{4AC' + 16}}$$

or

$$\sin(x\sqrt{A} + C_1') = \frac{An}{\sqrt{AC'}}$$

Solving for n, we obtain

$$n = \frac{\sqrt{AC' + 4D^2}}{A} \sin(x\sqrt{A} + C_1'$$

$$-\frac{2D}{A} = \frac{d^2y}{dx^2} = -Ay - Bx - Dx^2$$

tion of (13) a
ses, it is only
edundant ter
ts of integ

SOME OF THE PRINCIPAL CASES UNDER MIXED STRAINS.

I. *For the first case let the beam be fixed at one end, free at the other, and having a load* P *and pull* T *applied at the free end.*

Fig. 1 or 1a represents it. The first question arising is whether to apply (13) or (15). An inquiry will reveal the fact that the sign of T*y* determines which. The calculus demands that the

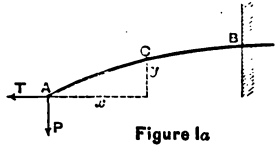


Figure Ia

ordinate and second differential coefficient have contrary signs when the curve is concave toward the axis of x. Hence in Fig. 1 they are contrary. Taking T positive when it is directed toward the beam, and negative when from it, we will have, for T pointing outward as in Fig 1 or 1a.

$$-\varepsilon I_1 \frac{d^2y}{dx^2} = Px - Ty = M$$

T being negative makes the sign of

and Ty alike.

For convenience let the subscript to I be dropped in the following equations, I being always regarded as the principal moment of inertia of the cross section of the beam. Then we have for the origin of moments at C, Fig. 1a, and of coördinates at A,

$$\varepsilon I \frac{d^2y}{dx^2} = Ty - Px = M \quad \ldots \quad (16).$$

To integrate this equation, we require (13) because the signs of Ty and Ay are alike, also of the second differential coefficients.

Comparing with (13)

$$A = \frac{T}{\varepsilon I} \quad B = -\frac{P}{\varepsilon I} \quad D = o \quad F = o$$

Hence

$$\frac{d^2y}{dx^2} - Ay + Bx = \frac{C_1 e^{nx}}{2} - \frac{C}{2C_1 A} e^{-nx}$$

or convenience $n = \sqrt{A} = \sqrt{\frac{T}{\varepsilon I}}$

n here differing from the n used in the integrations for (13) and (15).

The above equation for this case is given in *Strength of Materials*, by De Volson Wood, p. 300, 2d ed.

Let the length of the beam be l, and the distance of the point of application of T below the center of gravity of the end of the beam be h. Then if $x=o$ $y=h$, and

$$Ah=\frac{C_1}{2}-\frac{C}{2C_1A}$$

$\therefore$

$$\frac{C}{2C_1A}=\frac{C_1}{2}-Ah$$

and

$$Ay+Bx=\frac{C_1}{2}\left(e^{nx}-e^{-nx}\right)+Ahe^{-nx}$$

$$A\frac{dy}{dx}+B=\frac{C_1}{2}\left(e^{nx}+e^{-nx}\right)-Ahne^{-nx}$$

But

$$\frac{dy}{dx}=o \text{ for } x=l \text{ and}$$

$$B=\frac{C_1}{2}n\left(e^{nl}+e^{-nl}\right)-Ahne^{-nl}$$

The value of C_1 from this, introduced above, gives

Restoring the va
get

$$Ty - Px = -\left(\frac{P}{n} - T\right)$$

$$= M$$

This equation expre
between x and y for th
curve of the axis of the
say, of the curve AOB, F
maximum deflection of t
for $x = l$.

Also (17) is so arrange
moment of the applied
point, the first member b
as (16).

If $x = o$ $y = h$ as it

$$Ty_1 - Pl = -\frac{\frac{P}{n}\left(e^{nl} - e^{-nl}\right) - 2Th}{e^{nl} + e^{-nl}}$$

$$= M_{max}. \quad . \quad . \quad . \quad (18).$$

hen $h = o$,

$$y_1 - Pl = -\frac{P}{n}\left\{\frac{e^{nl} - e^{-nl}}{e^{nl} + e^{-nl}}\right\} = M_{max}. \quad (19).$$

ith h restored and $P = o$

$$Ty_1 = \frac{2Th}{e^{nl} + e^{-nl}} \quad . \quad . \quad . \quad . \quad . \quad (20).$$

Evidently the resultant of T and P ight be substituted, and be considered a single force acting obliquely. Conrsely any single force acting obliquely the end of the beam could be resolved to components corresponding with T id P.

If we imagine this resultant acting at e end of h, its prolongation would tersect the beam when h is downward. his point of intersection would evintly be a point of contrary flexure at hich the moment would be zero. The

Figure 1.

obtained from (19) is the same numerically for P + as for − which is evidently as it should be, while for T changed in sign it would be greater.

Hence, changing the sign of T, our equation of moments for this case becomes, as evident from Fig. 4, for the

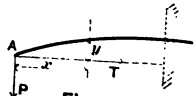


Figure 4.

origin of moments at C and of co-ordinates at A.

$$\varepsilon I\frac{d^2y}{dx^2}=-Ty-Px=M \quad \ldots \quad (21)$$

This may be integrated by aid of equation (15), as the signs are proper to it, where,

$$A=\frac{T}{\varepsilon I}=n^2 \quad B=\frac{P}{\varepsilon I} \quad D=o \quad F=o$$

Hence the integral is

$$-Ay-Bx=\sqrt{\frac{C}{A}}\sin(x\sqrt{A}+C_1)$$

$$=\sqrt{\frac{C}{A}}\sin(nx+C_1)$$

$$-Ay-Bx=$$

Differentiate and,

$$-A\frac{dy}{dx}-B=$$

But

$$\frac{dy}{dx}=$$

$$\therefore \quad B=Ahn(co$$

or

$$\text{Cot } C_1=\frac{}{Ah}$$

Hence

$$-Ay-Bx=-\frac{B}{n}\frac{s}{c}$$

$$-Ah\,(1+t$$

or, restoring the v
get the general

Here also we will find a point of contrary flexure, and of zero moment of flexure, where the resultant of P and T may, when prolonged, intersect the beam.

The maximum moment of flexure occurs where $x=l$ and $y=y_1$, which is

$$-Ty_1-Pl=-\frac{P}{n}\text{tang}nl-\frac{Th}{\cos nl}=M._{max} \quad . \quad . \quad (23).$$

when $h=o$

$$-Ty_1-Pl=-\frac{P}{n}\text{tang}nl=M._{max} \quad . \quad (24).$$

When h is restored, and $P=o$

$$-Ty_1=-\frac{Th}{\cos nl}=M \quad . \quad . \quad (25).$$

III. *Let the beam be conditioned as represented in Fig. 5, viz.: fixed at one*

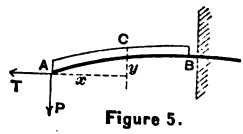


Figure 5.

end, free at the other, with a load **P** ***and a pull*** **T** ***at the free end, and a uniform load over its length.***

Let w = distributed load per unit length of beam.

Then the equation of moments will be

$$EI\frac{d^2y}{dx^2} = Py - Px - \frac{wx^2}{2} \quad . \quad . \quad . \quad (25)$$

Hence:

$$y = Ae^{nx} + Be^{-nx} + Dx^2 + \frac{C_1}{2}e^{nx} - \frac{C_1 e^{-nx}}{2C_1 n^2} - \frac{2D}{n^2} + \frac{2D^2}{C_1 n^4}e^{-nx}$$

where:

$$A = \frac{P}{EIn^2} \quad B = -\frac{P}{EI} \quad D = -\frac{w}{2EI}$$

To determine the constants of integration make x and y zero for the point A. This gives

$$0 = \frac{C_1}{2C_1 n^2} - \frac{C_1}{2} - \frac{2D}{n^2} + \frac{2D^2}{C_1 n^4}$$

Introducing this, differentiating, and making $\frac{dy}{dx} = o$ for $x = l$ we get

$$B + 2Dl + \frac{2D}{n}\frac{e^{-nl}}{\left(\frac{e^{nl} + e^{-nl}}{2}\right)n}$$

which introduced, gives us the following general equation of moments, and of the curve of the axis of the beam, T P and w being restored,

$$Ty - Px - \frac{wx^2}{2} = -$$

$$\frac{P + wl + \frac{w}{n} e^{-nl}}{n} \left\{ \frac{e^{nx} - e^{-nx}}{e^{nl} + e^{-nl}} \right\} + \frac{w}{n^2}\left(1 - e^{-nx}\right) \quad . \quad . \quad . \quad . \quad (27).$$

In this equation the sign of P may be + or −; that is, P may act down or up. Under certain circumstances the beam will be S shaped, with a point of contra flexure, or of zero moment.

To ascertain this point place the second member $= o$ and solve for x.

The maximum moment is evidently obtained by making $x = l$. The equation then will reduce to

$$Ty_1 - Pl - \frac{wl^2}{2} = -\frac{P + wl}{n} \left\{ \frac{e^{nl} - e^{-nl}}{e^{nl} + e^{-nl}} \right\} +$$

$$\frac{w}{n^2} \left\{ 1 - \frac{2}{e^{nl} + e^{-nl}} \right\} = M_{max} \quad . \quad (28).$$

If $w=o$ this equation reduces to (2), as it evidently should. Also P may be made zero, leaving simply the uniform load and T.

IV. Let the conditions be as in Fig. 1, except reverse T, *as in Fig. 4.*

Then

$$\frac{d^2y}{dx^2} = -Ty - Px - \frac{wx^2}{2} \quad \ldots \quad (29).$$

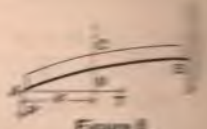


Figure 4.

Applying equation (15) for integrating this

$$-Ay - Bx - D = \frac{\sqrt{AC+4D^2}}{A} \cdot \sin(ax+C_1) - \frac{2D}{A}$$

where $A=\frac{T}{EI}$ $B=\frac{P}{EI}$ $D=\frac{w}{2EI}$

[illegible] the constants of integra- [illegible] $=o$ for $y=o$, and eliminate [illegible] 1st term, 2d member.

Then make the tangent to the curve zero for $x=l$, and eliminate C_1, as was done in case II., we then get

$$-Ty-Px-\frac{wx^2}{2}=-\frac{P+wl}{n}\frac{\sin nx}{\cos nl}-\frac{w}{n^2}(1-\cos.nx-\sin.nx\,\mathrm{tang}\,nl)=M \quad (30).$$

The load P may be + or −.

Making $x=l$ for the maximum moment, we get, by reduction,

$$-Ty_1-Pl-\frac{wl^2}{2}=-\left(\frac{P+wl}{n}-\frac{w}{n^2}\mathrm{tang}\frac{nl}{2}\right)\mathrm{tang}\,nl=M_{max} \quad . \quad . \quad (31).$$

If $w=o$, this expression reduces to (24), as it obviously should. If $P=o$ we have simply the uniform load w, and the end thrust T.

V. *Let the beam be supported at both ends, carry a uniformly distributed load, and have a pull applied at both ends.*

The conditions are obviously equivalent in the two parts of Fig. 7, the latter being already solved in equations (27) and (28).

The equation of moments for this figure is

$$\varepsilon I\frac{d^2y}{dx^2} = Ty - \frac{wlx}{2} + \frac{wx^2}{2} \quad . \quad (32).$$

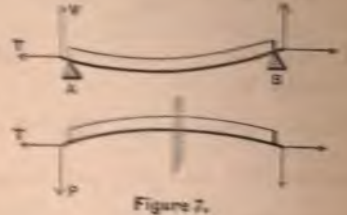


Figure 7.

since $V = \frac{wl}{2}$. Comparing with (27) we find the 1st member changed to (32) by changing P, $-w$ and l there, to $\frac{wl}{2}$, w and $\frac{l}{2}$ here. By making the corresponding changes in 2d member of (27) we obtain for the 2d member of (32),

$$+ w\left\{ \frac{e^{n\left(\frac{l}{2}-x\right)} + e^{-n\left(\frac{l}{2}-x\right)}}{e^{n\frac{l}{2}} + e^{-n\frac{l}{2}}} - 1 \right\} \quad (32).$$

length, AB.

For the maximum moment $x=\frac{l}{2}$ and

$$\mathrm{T}y_1-\frac{wl^2}{8}=\frac{w}{n^2}\left\{\frac{2}{e^{n\frac{l}{2}}+e^{-n\frac{l}{2}}}-1\right\}$$

$$\mathrm{M}_{\text{max.}} \quad . \quad . \quad . \quad (33).$$

VI. *Reverse* T; *otherwise take the case as in* V.

Hence reverse w, and take $\mathrm{P}=\frac{wl}{2}$ in equations (30) and (31) and change l to $\frac{l}{2}$ and we obtain

$$-\mathrm{T}y-\frac{wlx}{2}+\frac{wx^2}{2}=$$

$$\frac{w}{n^2}\left(1-\cos nx-\sin nx \operatorname{tang} n\frac{l}{2}\right)=\mathrm{M} \quad (34).$$

l being the whole length AB, and for the maximum moment $x=\frac{l}{2}$. $\therefore$

$$\therefore \quad -\mathrm{T}y_1-\frac{wl^2}{8}=-\frac{w}{n^2}\operatorname{tang} n\frac{l}{4}\operatorname{tang} n\frac{l}{2}$$

$$\mathrm{M}_{\text{max}} \quad . \quad . \quad . \quad (35).$$

VII. *Let the beam be supported at its ends, with a load* Q *placed at equal dis-*

tances from the ends, and have a tension T *applied.*

Figure 8.

1st. Then for the part AC, the moments will be

$$\varepsilon I\frac{d^2y}{dx^2}=Ty-Px=Ty-Qx \quad . \quad (36),$$

comparing with eq. (13) we get

$$Ay+Bx=\frac{C_1}{2}e^{nx}-\frac{C}{2C_1A}e^{-nx} \quad . \quad (13).$$

where

$$A=\frac{T}{\varepsilon I}=n^2 \quad B=-\frac{P}{\varepsilon I} \text{ and } P=Q \quad . \quad (37).$$

For $x=o$, $y=o$ and

$$\frac{C}{2C_1A}=\frac{C_1}{2}$$

$$Ay+Bx=\frac{C_1}{2}\left(e^{nx}-e^{-nx}\right)$$

Differentiate and let $\frac{dy}{dx}=\text{tang } i$ for $x=a$ -EG, and

$$\frac{C_1}{2}=\frac{A\,\text{tang}\,i+B}{\sqrt{A}\left(e^{na}+e^{-na}\right)}$$

$$\therefore\quad Ay+Bx=\frac{A\,\text{tang}\,i+B}{\sqrt{A}}\left\{\frac{e^{nx}-e^{-nx}}{e^{na}+e^{-na}}\right\} \quad \ldots\ldots (39).$$

2d. For the part CB, the moments will be

$$\varepsilon I\frac{d^2y}{dx^2}=Ty+Q(x-a)-Px=Ty-Qa$$

where

$$A=\frac{T}{\varepsilon I}=n^2 \text{ and } F=-\frac{Qa}{\varepsilon I} \quad \ldots (40).$$

$$\therefore\quad Ay+F=\frac{C'}{2}e^{nx}-\frac{C_2}{2C'A}e^{-nx} \quad \ldots (13).$$

Now differentiate this and make $\frac{dy}{dx}=o$ for $x=\frac{l}{2}$ and

$$\frac{C_2}{2C'A}=-\frac{C'}{2}e^{nl}$$

Put this back in the same equation and make $\frac{dy}{dx}=\text{tang}\ i$ for $x=a$, and

$$A\,\text{tang}\,i=\frac{C'}{2}\left(e^{na}-e^{n(l-a)}\right)\sqrt{A}$$

Now making the
two equations, equa
point C, where $x=a$
that $Ba=F$, (39)=(4
ion by aid of which
value of tang i, put
gives the required
moments on the p
spectively.

Placing the paren
equal L, and of (41)

$$A \text{ tang} i = -$$

The maximum mo

P and Q, is least at C and E, for the part CE. As to AC, the moment is greatest at C.

Hence (39) or (41) will give the max. moment which is at $x=a$

$$Ty_1 - Qa = \frac{Q}{2n} \cdot \frac{e^{n\frac{l}{2}} - e^{-n\frac{l}{2}} + e^{-n\left(\frac{l}{2}-2a\right)} - e^{n\left(\frac{l}{2}-2a\right)}}{e^{n\frac{l}{2}} + e^{-n\frac{l}{2}}} \quad (43).$$

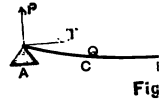
$$= M_{max.}$$

If we make $a=o$ the moment (43) reduces to zero, as it evidently should, as it leaves the beam without load. Again, if $a=\frac{l}{2}$, the expression reduces to the same form as (19), P and l there, being equivalent to Q and $\frac{l}{2}$ here, a fact that verifies all complex and extended reductions of this case.

VIII. *Let this beam be conditioned the same as in* VII., *except reverse* T, *as shown in Fig. 9.*

where

$$A=\frac{T}{\varepsilon I}=n^2$$



Fig

Hence 1st by (15),

for one constant, a

$=AC=$ EG for th

$-Ay-Bx=-(\sqrt{}$

Applying (15), first making $\frac{dy}{dx} = o$ for $x = \frac{l}{2}$ for one constant; then making the same equation $=$ tang i for $x = a$, for the other, we get

$$-Ay - F = -\frac{\sin\left(nx + \frac{\pi}{2} - n\frac{l}{2}\right)}{\cos\left(na + \frac{\pi}{2} - n\frac{l}{2}\right)}\sqrt{A}\,\text{tang}\,i \quad . \quad . \quad . \quad (46).$$

Then equating the ordinates y_1 from (45) and (46) for $x = a$, observing that $Ba = F$, we get the expression containing tang. i. Substituting it in (46) we get the general expression of moments for the part CE, also the equation of the curve of the axis of that part,

$$-Ty - Qa = -\frac{Q}{n}$$

$$\text{tang}na.\sin\left(nx + \frac{\pi}{2} - nl\right)$$

$$\sin\left\{\frac{\pi}{2} - n\left(\frac{l}{2} - a\right)\right\} - \tan na.\cos\left\{\frac{\pi}{2} - n\left(\frac{l}{2} - a\right)\right\}$$

$$= M \quad . \quad . \quad . \quad (47).$$

tang i, for $x=a$ for the other, and reducing we get

$$Ty-Px=\frac{T.\text{tang}\,i-P}{n}\left\{\frac{e^{nx}-e^{-nx}}{e^{na}+e^{-na}}\right\} \quad (50).$$

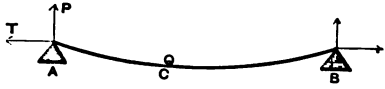


Figure 10.

From the figure $Pl=Q(l-a)$

$$\therefore \qquad P=Q\left(1-\frac{a}{l}\right) \quad . \quad . \quad . \quad (51).$$

If in (50) we make $a=\frac{l}{2}$ we have tang $i=o$, and introducing P, (50) reduces to the same as (19) and (43) for like values of a, and P.

2d. For the part BC with some reduction,

$$\varepsilon I\frac{d^2y}{dx^2}=Ty+\frac{Qa}{l}x-Qa$$

Applying (13) again, observing that for $x=l$ $y=o$, and $\frac{dy}{dx}=$tang i, for $x=a$, and reducing we get

This equation reduces
50), 51) and 52) for

by making $x = \frac{1}{2}$, exp

the negative sign, as in
moment at C should ap

Now make $x = 0$, and
52). This determines
be restored to 50) and
or for equations of the
of the parts of the beam
ditions of the beam are
termined.

If we make the paren

In seeking the maximum moment we suspect that when Q is near A, or B at a particular point, the combined moment of T and P may possibly have a maximum between Q and the remote end, because that due to T varies as the ordinate y and P with x. But the mathematical test applied to (52) develops no such max. Hence the max. is always at the load Q, and can readily be found at $x=a$.

X. *Let the conditions be the same as in* IX. *except reverse* T, *or put the beam under compression, as in Fig. 11.*

Figure 11.

1st. For part AC

$$\varepsilon I\frac{d^2y}{dx^2}=-Ty-Px$$

where

$$P=Q\frac{l-a}{l}$$

$$-Ty-(P-Q)x-Qa=-\frac{\frac{Q}{n}}{1+\tang na\cot n(l-a)}\cdot\frac{\sin n(l-x)}{\cos n(l-a)} \quad (57).$$

which are equations of the axial curves and of moments.

These reduce to the same as (49) for $x=a=\frac{l}{2}$, also to (24) for proper value of l. Thus (56) becomes

$$\frac{Q}{2n}\tang n\frac{l}{2}=M_{max} \quad . \quad (58).$$

the same as (48), observing that Q in (48) equals 2Q in the above.

In looking for the max. moment, we observe that when Q is near A, the moment will without question increase from A to C, because the moment arms of both T and P increase together. Also the moment of the reaction of the support B will increase from B to C. But the moment of T is greatest where the curve of the axis of the beam is lowest, and this is between Q and B. Now when T is great and P small, particularly when T

is sufficient to hold the beam in the curved form, the maximum is plainly seen to be somewhat removed from Q towards the middle of beam, but in no case beyond the middle.

The 2d member of equation (57) is the expression for the moment of flexure of BC. Hence the moment varies along BC as the ordinates of a sinusoid the max. value being where sin. $n(l-x)=1$, or where $n(l-x')=\frac{\pi}{2}$. Hence

$$\therefore \qquad \frac{x'}{l}=1-\frac{\pi}{2nl}$$

is the fractional part of the beam's length at which the moment *may* be a maximum, and where it will exist, in case BC has a less value. At $x=l$, or at B, the sinusoid starts, and overspans the beam length AB, as appears from the fact that when $x=o$, the value of (57) is still considerable, and positive. To find the ringing points of the curve,

$$\sin n(l-x)=o$$

giving

$$n(l-x)=o \text{ or} =\pi$$

Hence

$$x=l, \text{ or} =l-\frac{\pi}{n} \quad . \quad . \quad . \quad (59).$$

and the sinusoid is BED, for the beam AB, Fig. 12. The max. is at H when

$$AH=l-\frac{\pi}{2n} \quad . \quad . \quad . \quad (60).$$

equations (59) and (60) show that BD is twice BH. Suppose $n=.01$ and $l=200$. Then AH=43. and AD=−114.

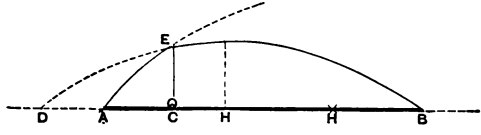


Figure 12.

The figure shows how the max. moment may occur at H when Q is at C, the curve AE being the sinusoid of eq. (56).

Similarly it may be shown that another point H′ exists where BH′=AH.

Now when Q is placed between H

max. moment is at Q. but it
H when Q is outside those

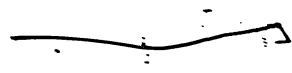


Figure 3.

$$\frac{\sqrt{C}}{n}=-\frac{F}{\sin C_1}$$

and for $\frac{dy}{dx}=o$ for $x=o$ we get

$$\cot C_1=\frac{B}{nF}$$

Hence

$$-Ay-Bx-F=-\frac{B}{n}\sin nx$$
$$-F\cos nx \ . \ (61).$$

and

$$-A \text{ tang } i-B=-B\cos nx+nF\sin nx$$

2d, For the part BC

$$\varepsilon I\frac{d^2y}{dx^2}=-Ty-P(l-x)$$
$$=-Ty+Px-Pl$$

Hence by (15)

$$-Ay-B'x-F'=\frac{\sqrt{C'}}{n}\sin(nx+C_1')$$

where

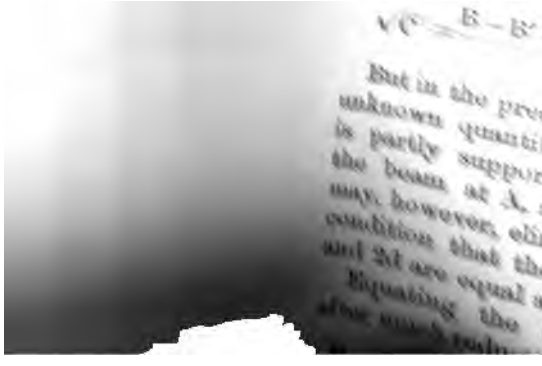
$$A=\frac{T}{\varepsilon I}=n^2,\ B'=-\frac{P}{\varepsilon I},\ F'=\frac{Pl}{\varepsilon I}.$$

also

$$B'l+F'=o, B-B'=\frac{Q}{\varepsilon I},\ a(B-B')=(F'-F)$$

For $x=l$, $y=o$ and

and



If $a=l$, $P=Q$; and if $a=o$, $P=o$; which are obviously correct.

The equation of the curve of the axis of the beam for AC, also the moment of flexure, will be

$$-Ty-Q\Big(q(l-x)-(a-x)\Big)=$$

$$-Q\Big((ql-a)\cos nx-\frac{q-1}{n}\sin nx\Big) \quad (64).$$

Similarly we have for BC

$$-'1\,y-Qq(l-x)=$$

$$-\frac{Q}{n}\Big(1-(1-q)\cos na+n(ql-a)\sin na\Big)$$

$$\frac{\sin n(l-x)}{\cos n(l-a)} \quad . \quad . \quad . \quad (65).$$

An inspection of the figure shows that here will always be a point of contra flexure in the beam, or where the moment of flexure will be zero.

Trying (64) for this point, placing the d number $= o$, we obtain

$$(1-q)\,tang\ nx=n(a-ql)$$

$$tang\ nx=na\frac{\left(1-q\frac{l}{a}\right)}{1-q}$$

these quantities are sma
small the tangent approx
when the fraction to na
which makes x less than

Trying (65), the 2d me
$\sin n(l-x)=o$, or $n(l-x)=$
But $x=l$ when $n(l-x)=o$

$$x=l-\frac{\pi}{n} \text{ when } n$$

But according to eq. (59
x in (65) has no value with
the beam for a varying fro

Hence we conclude that
contrary flexure is always
load O and th

if we ignore a. But considering a, it appears that the max. moment of flexure is always at Q, except when a is $=$, or less than x in (60); conditions identical with case X, as regards the one point H, Fig. 12.

Either member of (64) makes the moment at A where x and $y=o$

$$=-Q(ql-a) \quad . \quad . \quad . \quad (66).$$

But where $x=a$, the other limit of x for this expression, the moment is

$$-Ty-Qq(l-a)=$$

$$-\frac{Q}{n}\Big(n(ql-a)\cos na-(q-1)\sin na \quad (67),$$

and it is difficult to tell which is the greater, except in particular cases.

An inspection of figure 13 shows that if the compression T can be great enough to hold the beam in a curve while Q is zero, the force P will be reversed. Now, if Q be given some real positive value increasing from zero up, it will eventually make $P=o$. But by (63), when $P=o$, $q=o$. Hence to find the relation between T and Q for this, make $q=o$ in (64) and (65.)

XII. *Take the conditions as in* [illegible] *except change* **T** *to tension.*

Figure 14.

1st. For the part AC, origin of co- dinates at A, and y positive downward

$$EI\frac{d^2y}{dx^2} = +Ty + Q(a-x) - P(l-x)$$
$$= Py - (Q-P)x + (Qa - Pl)$$

Integrating by aid of (13) and determ ining the constants by the conditions $=o$, for $y=o$, and $\frac{dy}{dx}=o$ for $x=o$ we ge an equation of the curve of the axis of the beam AC, and also the moment of flexure

$$Ty - (Q-P)x - (Qa - Pl) =$$
$$\frac{Qa-Pl}{2}\left(e^{nx}+e^{-nx}\right) -$$
$$\frac{P}{2}\left(e^{nx}-e^{-nx}\right) = M \quad \ldots \quad (68).$$

But P is here an unknown quantity, and is to be found as it was in XI.

2d. For BC

$$\varepsilon I\frac{d^2y}{dx^2}=Ty-P(l-x)$$
$$=Ty+Px-Pl$$

which may be integrated by (13), for $x=l$, $y=o$; giving value to one constant. Then place tang i, for the point C equal the same from (68), making known the second constant. Then equate the ordinates y for the point C, by which to determine P. The work is too tedious to detail here, but the result is

$$\frac{P}{Q}=\frac{e^{-nl}-e^{nl}-e^{-n(l-a)}+e^{n(l-a)}+na\left(e^{nl}+e^{-nl}\right)}{e^{-nl}-e^{nl}+nl\left(e^{nl}+e^{-nl}\right)} \quad . \quad . \quad . \quad (69).$$

When $a=l$ $P=Q$, and when $a=o$, $P=0$; which are correct for these points and verify the result.

The maximum moment of strain for the whole beam, and under all conditions is probably at A, as it is at this point for

counter to that of Q, is g
the deflection is greatest, c
diate points.

XIII. *Let the beam be*
ends, be under tension T *an*
Q at the middle, and a un
tributed load.

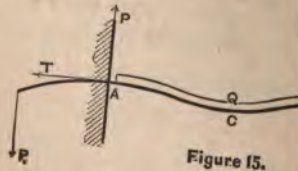


Figure 15.

ing that for $x=o$, $y=o$, and that $\frac{dy}{dx}=o$, for $x-o$ and also for $x=\frac{l}{2}$; and we get, after reduction, the general equation of moments and of the axis of beam

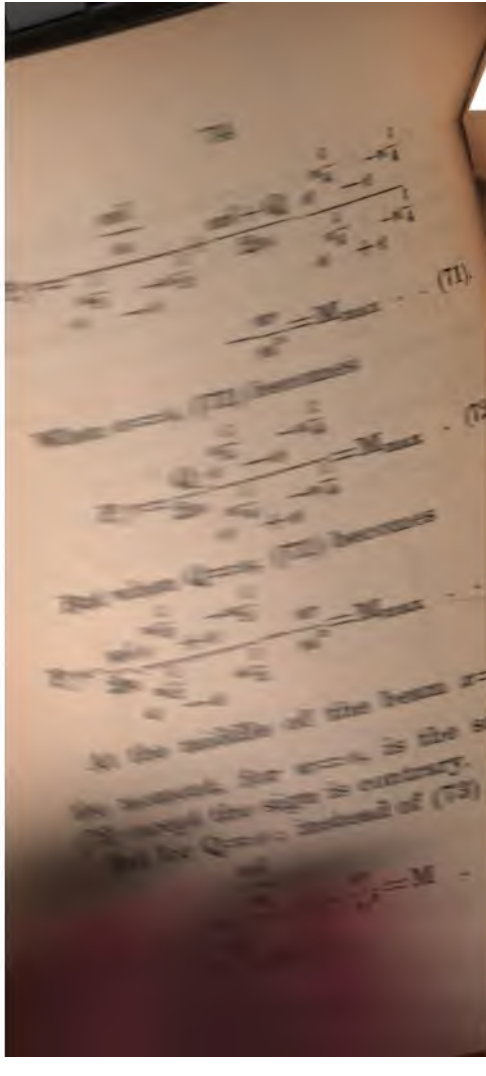
$$Ty+\frac{wx^2}{2}-\frac{wl+Q}{2}x+P_1r=\frac{wl+Q}{4n}$$

$$\left(\begin{matrix} e^{-nx} & e^{nx} \\ -e & -e \end{matrix} - \frac{2-\frac{2wl}{wl+Q}-e^{\frac{nl}{2}}-e^{-n\frac{l}{2}}}{e^{n\frac{l}{2}}-e^{-n\frac{l}{2}}} \left(\begin{matrix} e^{nx} & e^{-nx} \\ +e & \end{matrix}\right)\right)$$

$$-\frac{w}{n^2}=M \quad . \quad . \quad . \quad (70).$$

In case the load is only Q, make $w=o$, or if we have only the uniform load, make $Q=o$ and the resulting expressions will give the moment of flexure at any point in the length of the beam.

At the point A, the moment is a max. and found by making $x=o$, and $y=o$. Hence, as P_1r=moment at A, we have,



The four last equations give the moments most likely to be needed, and the value of $P_1 r$ put in (70) gives us the equation of the curve of AQ and BQ is like it.

XIV. *Take the case as in* XIII., *except change* T *to compression.*

$$\varepsilon I \frac{d^2y}{dx^2} = -Ty - Px + \frac{wx^2}{2} + P_1(r+x)$$

$$= -Ty - (P - P_1)x + \frac{wx^2}{2} + P_1 r$$

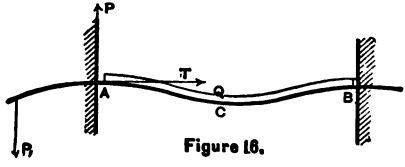


Figure 16.

Integrating by aid of (15), and observing that for $x=o$, $y=o$; and that $\frac{dy}{dx} = o$ for $x=o$ and for $x=\frac{l}{2}$; we obtain

$$-Ty - \frac{wl+Q}{2}x + \frac{w}{2}x^2 + P_1 r =$$

$$+\frac{Q}{2n}\frac{\cos nx}{\sin n\frac{l}{2}}+\frac{w}{n^2}=$$

At A, x and $y=o$, and

$$P_1r=-\frac{wl+Q}{2n}\cot n\frac{l}{2}+$$

$$\frac{Q}{2n}\frac{1}{\sin n\frac{l}{2}}+\frac{w}{n^2}=M_m$$

When $w=o$ (76), becomes

$$\frac{Q}{2n}\tan g\ n\frac{l}{4}=M_{max}\quad .$$

essentially the same as (58).

But when $Q=o$ in (76), we

$$P_1r=-\frac{wl}{2n}\cot n\frac{l}{2}+\frac{w}{n^2}=M_{max}$$

This is the same

$$-\frac{wl}{2n \sin n\frac{l}{2}}+\frac{w}{n^3}=M \quad . \quad . \quad . \quad (79).$$

Thus the most needed moments are obtained, and the equation of the elastic axis of the beam is fully given, P_1r, in (75), being made known by (76).

Strength of Wrought-Iron Bridge Members.

II. — Practical Formulas for Beams, Struts, Columns and Semi-Columns.— Extended Comparison of Various Formulas with Experiment.

General Applicability.

All expressions for the moment M, from equation (17) to (79) inclusive are applicable in equations (6) and (8), the former giving the strain t_1, due to the bending moment alone, and the latter the total or max. strain t, in the section considered. Any section throughout the length of the beam may be examined for strain, but usually the section under maximum strain is the one to be attended to.

It is to be observed that the strain due to M, is not in any case of the existence of T to be taken as the only strain. That due to T directly, according to (7), is to

be included. For instan
contrary flexure are usual
bearing sections which ar
strain except "transverse
all cases of points of m
mentioned above there wi
transverse shear, but dire
tension or compression al
resultant stress will in the
equal in direction.

It is to be observed that
been universally employed
sion and compression. T
for convenience. In s
proper distinction is to be

In cases where T is com
above expressions for M,
be regarded as acting part
and partly like a column
vary the intensity of act
almost purely of the first
the second kind.

duces to $\frac{0}{0}$ for T=o. But let the tangent be developed into a series, thus:

$$M_{max} = \frac{Q}{2n}\text{tang } n\frac{l}{4} =$$

$$\frac{Q}{2n}\left\{n\frac{l}{4}+\frac{n^3}{3}\left(\frac{l}{4}\right)^3+\frac{2n^5}{15}\left(\frac{l}{4}\right)^5+\&c\right\}$$

$$=\frac{Q}{2}\left\{\frac{l}{4}+\frac{n^2}{3}\left(\frac{l}{4}\right)^3+\&c\right\} \quad . \quad . \quad (80).$$

or

$$M_{max}=\frac{Ql}{8} \text{ for } n=o \quad . \quad . \quad . \quad (81).$$

This is the same expression for the maximum moment of flexure as we find given in works on the strength of beams for the case of a beam fixed at the ends and a load Q at the middle; conditions identical with (81) where T=o.

Similarly, (78), by expanding the $\cot n\frac{l}{2}$ gives

$$M_{max}=P_1 r=-\frac{w}{n^2}+\frac{wl^2}{12}+\frac{wn^2l^4}{720}+\&c+\frac{w}{n^2}$$

$$=+\frac{wl^2}{12} \text{ for } n=o \quad . \quad . \quad . \quad (82).$$

o. But as we propose that T
be the predominating force ap-
to the piece, then $T y_1$ cannot be
less $y_1 = o$, and the latter cannot be
l since T may produce considera-
lection to one side or the other.
tically there would be no deflec-
ut we know that practically there
be a very appreciable deflection for
of ten to twenty or more diame-
length before rupture; hence, we
reat the case as providing for a
ion, y_1.
ce it appears that the denominator
e infinite, which requires that

$$nl = \frac{\pi}{2}$$

ce $n^2 = T \div \varepsilon I$

$$\frac{T}{\varepsilon I} = \frac{\pi^2}{4l^2} \quad . \quad . \quad . \quad (83).$$

olumn fixed at one end and free
other, as in Fig. 17.
same result is obtained from (31).
xpression (83) is given in several
and produced in different direct

rounded, flat, or otherwise; that is, whether the end of the column is fixed or hinged, &c. The manner of holding the end, however, is not a matter of indifference, equation (49) showing that (84) is for rounded or hinged ends as shown in Fig. 18.

Figure 18.

Equation (77) or (78), for Q or $w=o$ and T not, requires that

$$\text{tang } n\frac{l}{4}=\infty \text{ or } n\frac{l}{4}=\frac{\pi}{2}$$

or that

$$\cot n\frac{l}{2}=\infty \text{ or } n\frac{l}{2}=\pi$$

either of which gives

$$\frac{T}{\varepsilon I}=\frac{4\pi^2}{l^2} \quad . \quad . \quad . \quad (85).$$

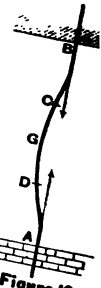


Figure 19.

This expression is not
published works as applyin
with fixed ends. An artic
merit which appeared in VAN
ENG. MAG., Vol. 17, p. 257
above equatio

also (87) with statement of end conditions.

To obtain the expression for a column having one end fixed and the other end rounded, we naturally turn to case XI, above. But the deduction of a satisfactory expression from this is difficult. However, reasoning from analogy, compare Figs. 18 and 19 with their expressions. Thus,

$$\left.\begin{array}{l}\text{For Fig. 18, } nl=\pi \\ \text{For Fig. 19, } nl=2\pi\end{array}\right\} \quad . \quad . \quad . \quad (86).$$

Observing that the curves are sinusoids—see eqs. (22) for $h=o$ and (47.)—we find AB, Fig. 18, like c D. Fig. 19, both being complete flattened sinusoids. But the middle of the beam at G, Fig. 19, is parallel to the same at A or B; from which we conclude that

$$AD+BC+CD=2 \text{ sinusoids.}$$

Hence the number of sinusoids in the curves Figs. 18 and 19 agree with the coefficients to π in the respective expressions (86).

for the case of columns having one end flat or fixed, and the other rounded or pointed.

The conditions are shown in Fig. 20,

Figure 20.

where A is fixed, and B rounded or hinged.

These formulas. (83) and (84), have been proposed for use by some writers, without modification, though apparently with much hesitation and doubt: others have declared them wholly unfit. It appears that Hodgkinson compared (84) in the form

$$T=\varepsilon\pi^2\frac{I}{l^2}=\frac{\varepsilon\pi^2}{64}\cdot\frac{d^4}{l^2}$$

where

$$I=\frac{\pi d^4}{64}$$

with his experimental results, but finding it discordant, only used it as suggestive of form for an expression for columns, thus:

$$W=C\frac{d^n}{l^m} \quad . \quad . \quad . \quad (88).$$

where W = wt. sustained, d = diameter, l length; and with C, m and n, constants to be determined by experiment. The differences made manifest by Hodgkinson's experiments between (84) and (88), as to the values of m and n, are such, except by further inquiry, as to destroy all confidence in (84) as a formula to use directly in calculating the strength of columns. The same is true of (85) and (87.) This is unfortunate as regards one important fact, viz: (84) contains the
f inertia of the cross section
merely the diameter, so that
applied to any form of sec-

tion, such as round, square, oblong, I and H sections &c., while (88) must be prepared for each characteristic section. Also, (84) contains one important factor, ε, which qualifies that formula for at least this one property of the material going into the column. But evidently there is, at least, one other property which is equally important, viz: the crushing resistance; or, if preferred, the elastic limit, neither of which appears in (84). Hence, though (84) has some valuable features, it must be pronounced sadly deficient as a general formula, except possibly for very long columns, such as would fail by simply springing out, without breaking. But evidently, columns relatively short, though longer than "blocks," and such as would fail partly by crushing and partly by springing, should have a formula containing both the coefficient of elasticity ε, and some modulus of resistance, such as t, for compression or tension, usually the former. Again still shorter columns or "blocks" would only require the modulus of crushing resistance.

Let Fig. 21 represent a portion
column under strain from the actio
T, the strain being partly compres
and partly flexural.

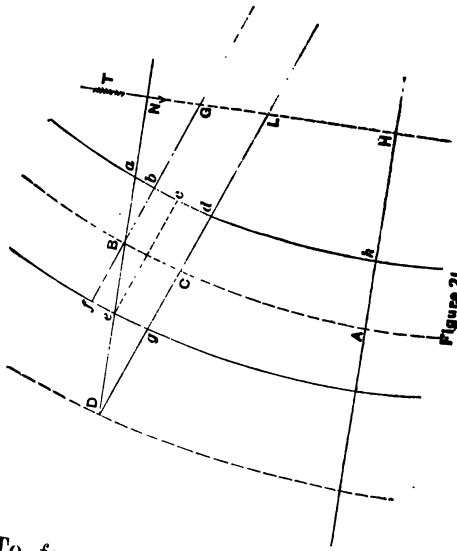


Figure 21.

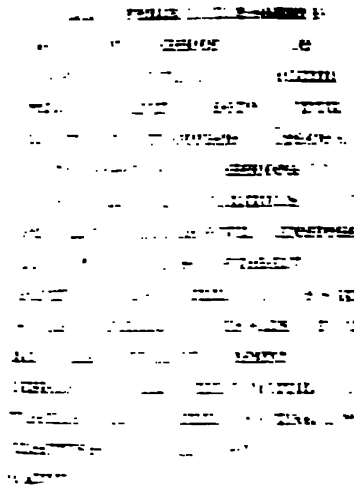
To formulate the relations between
ions, suppose AB, Fig. 21, to be
portion of a column near its

middle, which is sprung out of its original axial position NGH, to AC, the original axial fiber HN being now shortened and deflected to AC. The section *dg* was before flexure on the line *ae*, the intersection at D being a point on the real neutral axis of the column. The distance along DC to the intersection with AH, is the radius of curvature ρ. Draw BG parallel to DC, intersecting AH. The distance from this latter point to B is equal to ρ, because of the parallelism of BG and DC.

Then, according to (5), taking the line GH as the axial line of the column previous to flexure, and the line along which the compressive force T still acts, we will have the applied bending moment equal

$$\frac{\varepsilon I_1}{\rho} = Ty_1 \quad . \quad . \quad . \quad (89).$$

where I is the principal moment of inertia, or moment of intertia, for an axis perpendicular to the plane of the curve ACB, at the center of gravity B of the section considered; $y_1 = BN$, the moment



tinguish it as the maximum ordinate to the axial curve of the column; that ordinate being chosen because the moment of T is here greatest, and hence may denote the ordinate at the rupturing point of the column.

In (89) the 2d member is the applied moment, and is that which simply bends the column. The 1st member is the moment of resistance to this bending, and represented in the figure by the triangle B*ab* of compression, and B*ef* of tension.

To obtain another equation take the origin of moments at D on the real neutral axis. Then the total moment of resistance is

$$\frac{\varepsilon I}{\rho} = T\,(y_1 + BD) \quad . \quad . \quad . \quad (90).$$

where I is now the moment of inertia of the section at B for an axis at D; $(y_1 + BD)$ the moment arm of T; and ρ the same as in (89), as already stated. By a well-known formula, we have

$$I = I_1 + K.\overline{BD}^2$$

the ordinate y_1 let T be applied at E, g. 22, to a prism of unit's length. First, the distance, BE, = the *principal radius of gyration*, $= k$. The point B is center of gravity of the section at ED,

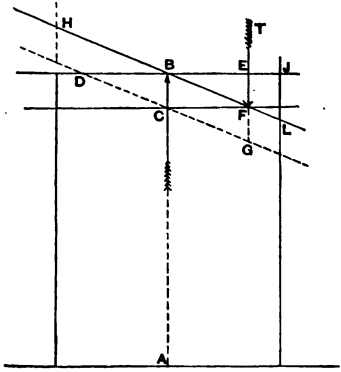


Figure 22.

top of the prism. Now, as T is applied E, let an equal and opposite force be plied at B, as shown. We thus have a ple acting to turn the top surface of prism around B from the line EBD to

FBH. If now we remove the counteractin
force at B, while T is kept constant, th
point, B, will fall by the compressi
action of T to some point C. Call BC=
This will be found =EF, when BE=
For we have, by taking moments at B,

$Tk=\frac{\varepsilon I}{\rho}$; also $I_1=Kk^2$, and $T=\varepsilon K\lambda$

combining,

$$\frac{T}{\lambda}\frac{k^2}{\rho}=Tk \text{ or } k=\rho\lambda$$

whence

$$\rho : 1 :: k : \lambda, \text{ or : BC}$$

Hence λ=BC=EF as proposed.

Applying T at any other point, we hav
a different result, for suppose it at
The compressive effect, BC, will be th
same for the same value of T as long
it is the only compressive force. Bu
the fall due to bending will be greate
even greater than JL, because the m
ment arm is greater.

Hence when a compressive force, T,
applied at the distance, k, from the ce
the cross section of a prism, th

displacement, EF, due to bending is exactly equal to the displacement, FG, or BC, due to direct compression.

Next suppose the neutral axis to be taken at a distance qk from B while T is applied at E. Then eq. (90) gives

$$T(k+qk)=\frac{\varepsilon I}{\rho}=\frac{\varepsilon I_1}{\rho}+\frac{\varepsilon}{\rho}K(qk)^2$$

or since $Tk=\frac{\varepsilon I_1}{\rho}$ . . . (92a).

$$Tqk=\frac{\varepsilon K}{\rho}q^2k^2$$

$$Tk=\frac{\varepsilon K}{\rho}qk^2 <= \text{or} > \frac{T}{\rho\lambda}k.qk$$

where $\rho\lambda=k$ when T is applied at E, and $T=\varepsilon K\lambda$

Hence

Tk=applied moment

<=or> moment of resistance.

To make the first and second members equal, we must put $q=1$; and hence we must also have the neutral axis at a distance, k, from the center, B, of the section, for the condition that the column is indifferent as to whether it deflects more

and

$$\therefore \qquad ad=\frac{t}{\varepsilon}=\frac{y_1+d_1}{\rho} \quad . \quad . \quad . \quad (93).$$

But combining

$$\frac{\varepsilon I_1}{\rho}=Ty_1 \quad . \quad . \quad . \quad (94).$$

with equations (84), (85), and (87), observing that I is the same as I_1 in (94), we get for columns

$$\left.\begin{array}{ll} \text{with round ends...} & \frac{1}{\rho}=y_1\frac{\pi}{l^2} \\ \text{with flat ends.....} & \frac{1}{\rho}=4y_1\frac{\pi^2}{l^2} \\ \text{with round and flat.} & \frac{1}{\rho}=\frac{9}{4}y_1\frac{\pi^2}{l^2} \\ \text{with fixed and free.} & \frac{1}{\rho}=\frac{1}{4}y_1\frac{\pi^2}{l^2} \end{array}\right\} \quad . \quad . \quad (95).$$

Combining these with (93), we obtain

$$\text{for round ends} \quad \frac{tl^2}{\varepsilon\pi^2}=y_1^2+d_1y_1$$

or, solving for y_1

$$y_1=\frac{d_1}{2}\left(\sqrt{1+\frac{4tl^2}{\varepsilon d_1^2\pi^2}}-1\right) \quad . \quad (96).$$

But by eq. (8), the total compression on the critical fiber *ah*, Fig. 21, is

$$t=Ty_1\frac{d_1}{I_1}+\frac{T}{K} \quad . \quad . \quad . \quad (96a).$$

We observe that in using the equations (84), (85) and (87), our object is to obtain a relation between ρ and y_1, as given by the curves of sines of the elastic axes of the respective columns, and not to employ them as the foundations for the general formulas, because the latter object is found in (93), (94) and (96a), which give the limiting ordinate, y_1 the moment of flexure, and the limiting strain respectively.

In these formulas t is the greatest admissible compression per square inch, or the modulus of crushing, for breaking loads, K, the sectional area of the column at the middle, d_1 the distance from the center of gravity of the section to the fiber which ruptures first, I_1 the moment of inertia for an axis at the center of gravity of the section, and at right angles to the plane of the axial curve of the deflected column; l, the length of the column, and ε, the coefficient of elasticity of the material. Also we have

$$\frac{K}{I_1} = \frac{1}{k^2}$$

ends rounded ..... $\frac{l}{d}=36.2$

ends flat..... $\frac{l}{d}=72.4$

ends round and flat.... $\frac{l}{d}=54.6$

For square, hollow and open-built columns, the ratios are still greater.

Hence, these last named formulas, previously rejected, now return to us again for application to very long columns. For this purpose, fortunately, the objection to these formulas, previously stated, of the absence of the modulus of crushing, t, has no weight, because failure occurs by springing and not by crushing.

Criterion for Long Column Formula.—Evidently this limit is to be found by placing the values of the load T as given by the two formulas equal to each other, and then solving for the

$$\text{limiting ratio}\frac{l}{d}$$

For rounded ends we have therefore

$$1+\frac{Ka_1}{2I_1}\left(\sqrt{1-}\right.$$

To avoid multipl
is desirable to obtai
the three conditio
ends flat, ends rou
oe done by aid o
(104). To adapt a

member we obser

To provide for t
radius of gyratio
stant ratio to eac
assume

$$\frac{d_1}{k}$$

an expression in which the first member, placed equal $\frac{T}{K}$, will give any one of the expressions (84), (85), or (87), by introducing proper values of α and β. Also the second member of which will give (97), (98) or (99), with like attention to α and β.

Clearing of fractions we obtain

$$\sqrt{1+\frac{atl^2}{\varepsilon\pi^2 d_1^2}}=\left(1-\frac{2}{\beta^2}\right)+\frac{atl^2}{2\varepsilon\pi^2 d_1^2}$$

Squaring, and solving for $\frac{l^2}{d_1^2}$, we obtain

$$\frac{l^2}{d_1^2}=\frac{4}{a\beta}\left(1+\frac{1}{\beta}\right)\frac{\varepsilon\pi^2}{t} \quad . \quad . \quad . \quad (106).$$

a formula which serves as the criterion of applicability for $\frac{T}{K}=$ first part; or= second part of (105), as a formula for extremely great or for ordinary lengths, respectively.

When the axis for the moment of inertia is an axis of symmetry for the cross section, we have $2d_1=d=$ whole diame-

ter of column. This is the thickness in this case in the direction of the plane of the curve of flexure of the column, and is the extreme thickness, whether the column be solid, hollow or open-built.

When the cross section is thus symmetrical, we have

$$\frac{l}{g d_1} = \frac{l}{d} = \pi \left\{ \frac{1}{a\beta} \left(1 + \frac{1}{\beta}\right) \frac{\varepsilon}{t} \right\}^{\frac{1}{2}} \quad . \quad (107)$$

As an example, in cylindrical columns, d_1 is the radius of the column, and k the radius of gyration.

$$\frac{d_1}{k} = 2 = \beta$$

and for flat ends,

$$a = 1$$

Taking $\varepsilon = 28{,}000{,}000$ and $t = 40{,}000$, we obtain from (106) or (107) the value 72.4, the same as given above.

FORMULA FOR END CONDITIONS.

us now determine a general ex- for the qualifying factor a, an on which shall duly qualify the formulas (105) for the

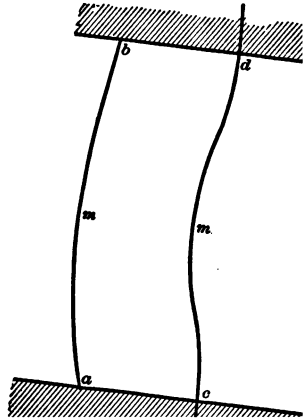
case of fixed ends, flat ends, or of rounded ends (pointed); or for pin ends with large pins under friction, or for flat ends which are narrower than the middle section, or for one of these forms at one end, and another at the other, &c.

From (97), (98), and (99), we already have

$$\left.\begin{array}{l}\text{For pointed ends}\ldots\ldots a=4,\ m=o\\ \text{For fixed ends}\ldots\ldots a=1,\ m=2\\ \text{For pointed and fixed.} a=\frac{1}{9}^{6},\ m=1\end{array}\right\}\ (108)$$

where m stands for the moment effect of the end conditions preventing lateral displacement of column at the middle. In Fig. 23, the pointed ends a and b offer no hindrance to the side deflection of the middle, m, of the column, hence in (108) m is zero. But at c and d, there is the stiffeners of the fixed ends, to prevent the middle from deflecting; and hence for this case in (108), $m=2$, similarly for the third part of Fig. 23, and of (108) $m=1$.

Fixed Ends.—Now if we consider the stiffness at a fixed end as equivalent to a


b
d
m
m.
a
c

$$p\frac{l}{2}=\mathrm{T}k'$$

where k' is different from k when the fixed end of column differs in size and form from the middle section. Designating the latter, or radius of gyration of the middle section by k_m, and of the end surfaces or sections by k_c and k_d, we will have the total counteracting moment

$$\left(\frac{p\frac{l}{2}}{\mathrm{T}k_m}\right)=\frac{\mathrm{T}k_c+\mathrm{T}k_d}{\mathrm{T}k_m}=\frac{k_c+k_d}{k_m} \quad . \quad (109).$$

an expression which admits of any size of end sections less than the middle section. Even the ends c and d may differ. Now we find by trial that a may be obtained from the expression

$$a=\frac{4}{\left(1+\frac{k_c+k_d}{2k_m}\right)^2} \quad . \quad . \quad . \quad (110).$$

for ends actually fixed.

To illustrate, if k_c and k_d are zero, $a=4$ as for pointed ends, thus producing (97) from (105) as regards qualifying with respect to a. Again put $k_c=k_d=$

$= k_m$ and $a = 1$, as for (98). Finally for $k_c = k_m$ and $k_d = o$, we have $a = 1\frac{5}{9}$ as required in (99).

But suppose the ends are only half as large as the middle and similar. Then $k_c = k_d = \frac{1}{2}k_m$ and a becomes $1\frac{5}{8}$, the same as for one end pointed and the other as large as the middle, and fixed.

To distinguish between actual fixed ends, and flat ends, it appears that in the former the column cannot fail at the end without a similar failure at the middle. According to the conclusion BD$=y$, in Fig. 22, there must be some tension in the section both at the middle and end of column. That is; the neutral axis is at D when rupture takes place, and outside of D there will be tension. Hence when the end sections are cut as in flat ends; that tension cannot act, and the column is weaker than for actual fixed ends. It appears therefore that the neutral axis should not approach the axis of the flat ends nearer than the outermost surface of the column. As the neutral axis is at D, where BD=the radius of gyration, k;

we may obtain in flat ends, the equivalent to fixed ends, by making the radius of gyration of the flat end surface of the column equal to d_1. This necessitates enlarged, or disc-like ends.

Equation (110) is supposed to meet the real case of fixed ends. To adapt it to the case of flat ends, it is only necessary to replace k_m by d_1 : because the same value of a would then be given by (110) thus modified for k_c and $k_d = d_1$, as would now be given by (110), as above, for k_c and $k_d = k_m$. That would make the radius of gyration of the flat end bearing $= d_1$ and the neutral axis could not approach the axis at the end of the column nearer than d_1. Then there would be no tendency to tension in the flat ends, as well as no power to resist it.

Again in built columns it is often the case that thickening pieces are laid on at the ends, thus making the area of the flat ends greater than the sectional area at the middle. From analogy with the usual relation of k and K in sections

Hence, to include th
tions of (110) now cons
that the equation (110)
ten.

$$a=\frac{4}{\left(1+\frac{k_c\sqrt{K_c}+k_d\sqrt{}}{2d_1\sqrt{K_m}}\right.}$$

The meaning of th
indicated in Fig. 23.

When the end surface
every way equal the mi
the column, the capitals
and disappear, also whe
actually fixed d_1 should
and thence (110a) return

When the flat ends
either in diameter

than that for fixed ends, which is evidently as it should be.

In no case should (110a) ever be used when the denominator gives a value greater than the denominator of (110).

Pin Bearings.—The pin bearing for the end mounting of a column is usually considered as acting in effect like a knife-edge terminal to the column. But it is evident at once that the friction of the pin must cause the column to act as though the end had some breadth which is appreciable, and hence not to be ignored. To provide for this, let r be the radius of the pin, and f the coefficient of friction.

Then, as the column bows to one side or the other, the point of bearing of the hole upon the pin will be thrown to one side a distance ox Fig. 24, which distance is greater or less according to the value of f. Now as is well known, the value of ox is

$$ox = fr.$$

By Fig. 23, this may be designated as

corresponds with T
fixed end. Hence,
we may write

$$\frac{\left(p_1 \frac{l}{2}\right)}{T k_m} = \frac{f}{-}$$

a quantity which is
(109) and may be inc
thus

$$a = \frac{}{\left(1 + \frac{k_c \sqrt{K_c} + k_a}{2 d_1 \sqrt{K}}\right.}$$

In bridges, the pi

If the column be a solid cylinder, 8 inches in diameter at the middle $k_m = 2''$; and observing that k_c and k_d are both zero here, we have

$$a = 3.16$$

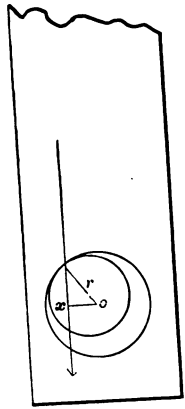


Figure 24.

As ordinarily treated, this would be taken as 4 instead of 3.16.

Relative Positions of Pin Bearings.— Now with respect to knife-edged *vs.* point-

ed ends, it is evident that when the knife edges are in one plane, and so oriented as to admit of deflection in the same direction as when the ends are pointed, the supporting power of the column is the same for either form of end. But when the knife edges are in a plane at right angles to the plane of deflection for points, the strength of the column is greater. This is with the understanding that the moment of inertia of the middle section is greater for an axis in one direction than in another. With a given cross-section of column the greatest strength with pin bearings is obtained by placing the knife edges parallel to the axis for that max. moment of inertia of that cross section. For deflections conceived by guides to be in the line of the knife edges, the strength is as for a column with flat ends. When a knife edge is placed at each end of a column, the strength is increased by placing the knife edges at right angles to each other. [illegible] this hypothesis it is difficult to [illegible] the real strength of the col-

umn, or to determine the direction of the deflection.

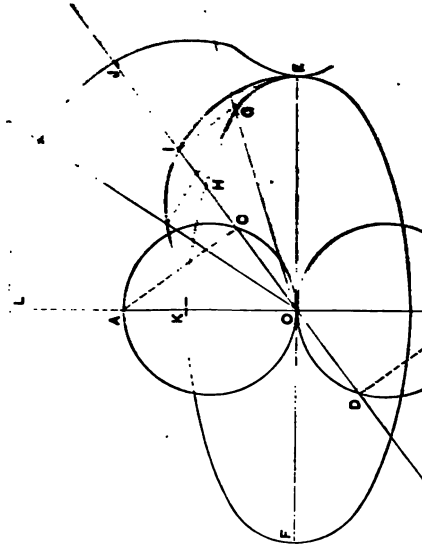
To investigate this let us first take the first part of eq. (105). Replacing β by its value we have for the supporting power per square inch

$$\frac{T}{K}=\frac{4\varepsilon\pi^2}{a}\cdot\frac{k^2}{l^2}=t$$

Let l be constant for a particular case. Then the strength is constant if $\frac{k^2}{a}$ is constant. If this ratio is variable the strength of the column is variable, for deflection in different directions as controlled by guides near the middle. For each position of the guides there may be noted a deflection, a radius of gyration, and a knife edge component.

According to considerations previously noted, the column is counteracted in its tendency to deflect by a moment Tk at the middle, and Tx, say, at the end; where x is the arm of the moment. Now for the middle cross section of the column; let the radius of gyration be

found for a number of different directi
within the circumference. Lay these
as in Fig. 25 where OE OG OH



values for the 4 respective di
Then trace the curve EG H
lius vector to this curve will

the value of k lying in the direction of that radius vector.

The point O represents the axis of the column. Let AB represent the knife edge bearing at one end of the column, its effective length being laid off to the same scale as k. The effective length should be less than the actual length, for the same reason that k is less than d_1; that is, the pin bearing will crush for some distance, corresponding to $d_1 - k$.

Now for a constrained deflection toward OA, lay off $x = OA$ as the moment arm. This added to k gives $OK + OA = OL$. Again for a constrained deflection toward OJ the knife-edge component of OA will be $OC = OA$ projected upon OJ. C is a point in a circle OCA giving all components. The curve EGHK should be treated in a similar way for finding the components. For instance project upon OJ by the tangent from the curve EGH giving I one point in the curve EIK. Any radius vector to this curve gives the k component for that direction. Now add OC and OI giving OJ.

J is another point in the curve EJL, a radius vector to which represents the resultant moment arm for the moment counteracting deflection in the direction of radius vector considered.

We should yet add the counteracting effect due to the opposite end of the column. Another curve outside of EJL including the three; the middle and the two ends. Then the minimum radius vector to that curve will be in the direction in which the column will deflect when unconstrained.

In practice it will be found that for almost every possible case, the minimum radius vector will lie in the direction of OA or OE so that the column will but rarely deflect diagonally. The minimum load value of the column will be found by computing for the direction of deflection above indicated. If the deflection takes in the direction of the pins, a is to be found from (111) by ignoring r and [illegible] and k_d as equal the respective [illegible]A.

[illegible]ife edges be at right angles,

the curve, Fig. 25, obtained will be found to have minimum ordinates in the directions of the pins or knife edges.

Fig. 25 applies especially for the long column formula. For shorter columns proper to the formula (101) we will probably find less decided differences between max. and min. radii vectors to Fig. 25 until for short blocks the differences are either indeterminate or indistinguishable.

Open Column Stays.—Open built columns are occasionally deficient in trussing or latticing. For instance, two 8″ channel bars may be joined by $\frac{3}{8}$″ plates 10″ square riveted on in such a way that the plane sides of the bars are back to back with an intervening space of 6 or 7 inches, the flanges being turned outward. Then if the plates are placed 3′ 10″ apart along the bars, the column thus constructed will probably be deficient in cross stays. In such case the column will fail to carry the load determined for it by the usual formula and

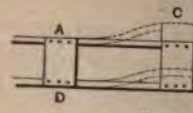


Figure 2

has given, the parts A
columns with fixed en
to find the length of
the same plates fully
be equivalent in stre
part AB with its defici
AB less the width of
length of equivalent
Then if the parts AB
ed as small columns w
have by applying the

the least moment of inertia of each of the parts AB DE &c. taken separately and added; and I_1 the least moment of inertia of the entire section of the column obtained in the usual way as such. In this section, the staying is not counted. There the square of the ratio obtained by dividing the equivalent full-stayed column length by the actual column length supposed insufficiently stayed is obtained by solving the above for l''^2 and dividing by l^2, giving

$$\left(\frac{l''}{l}\right)^2 = \frac{I_1}{I'}\left(\frac{l'}{l}\right) \quad . \quad . \quad . \quad (112).$$

This forms a co-efficient for l^2 in any of the column formulas. For instance, if $\frac{I_1}{I'}\left(\frac{l'}{l}\right)^2 = 3$, then in formulas (97), (98), or (99), &c., l^2 therein, should be multiplied by 3, when (112) is less than 1 it is not to be applied. This fact indicates that the column is sufficiently stayed.

Stay Spacing.—In the last statement we have the suggestion of a formula for determining the spacing in built columns.

That is to say, placing
of (112)=1, and solving
twice the greatest allo
tween stay points in ope
or,

Greatest admissible spa
points

$$\frac{l'}{2}=\frac{l}{2}\sqrt{\frac{I'}{I_1}} \quad . \quad .$$

Thus, if a column be br
bars L $\frac{1}{2}''\times 3''$, situated as
corners of a 12″ square in
long, the greatest inter-stay
about 16 inches. As an exampl
experiment, column No. 33
caren's report of experiments
C. E. gives the spacing o
column at 18 inches, and i
buckling between the lattices.
ing, according to the form
should not have exceeded abou
and hence it should have fail
served.

o the space in open colu
construction, it is evi

AB=l' is twice the space in Fig. 26, and in the first part of Fig. 27. But in the second part of the latter, AB=l' is to be counted as the space where the cross and diagonal staying together serves to fix the ends of the parts, AB.

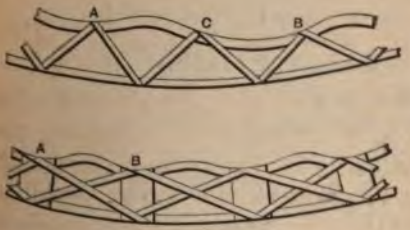


Figure 27.

PRACTICAL FORMULAS FOR COLUMNS.

Collecting the formulas which are useful in practice, we have:

For ordinary columns

$$\frac{T}{K}=\frac{t}{1+\frac{d_1^2}{2k^2}\left(\sqrt{1+\frac{atl'^2}{\varepsilon\pi^2 d_1^2}}-1\right)} \quad . \quad . \quad (114).$$

For very long columns

$$\frac{P}{A} = \frac{k}{l} \pi^2 \frac{E}{n^2} \quad . \quad .$$

Criterion for applying 114 or 11

$$\frac{l}{2k} = \frac{1}{n} \frac{\pi^2}{\sqrt{3}} \left(1 - \frac{1}{6}\right)^{\frac{1}{2}} \left(\frac{E}{f}\right)^{\frac{1}{2}}$$

$$n = \frac{l}{k}$$

For shaping to actual end co
see Fig. 23.
For flat ends.

$$a = \frac{4}{\left(1 + \frac{k_1 \sqrt{K_1} - k' \sqrt{K_2}}{2l \sqrt{K_m}} + f'\right.}$$

For *fixed* ends change d_1 to k_m

Co-efficient for l' in (114)
for columns deficient in staying

$$\left(\frac{l'}{l}\right)^2 = \frac{I_1}{I'} \left(\frac{l}{l}\right)^2 \quad .$$

Inter-stay spaces in open colum

$$\frac{l'}{2} = \frac{l}{2} \sqrt{\frac{I'}{I_1}} \quad . \quad .$$

where $l' = AB$ in Figs. 26 and 2

In these formulas,

=load borne by the column.

=compressive resistance per square inch corresponding to T.

=cross section of column, K_m at middle; K_c and K_d at ends, either when flat or fixed, and when flat to be taken as the flat-end surface. Only for suporting members.

=distance from the center of gravity of the section K_m to where the strain is t, and measured on the plane of the curve of flexure.

$=2d_1$ in symmetrical cross sections, and equal the diameter of the column reckoned parallel to plane of flexure.

=radius of gyration of the cross section, k_m, least radius at the middle of column; k_c and k_d radii for the end sections for fixed ends; or of the end surface for flat ends, and to include supporting members only.

= length of the column.

= coefficient of elasticity of the material composing the column.

column

I_1 = least m
cross s
taken i
and exc

I' = aggreg
ed by fi
ertia of
separat

$\frac{l}{2d_1}$ > than ir

$\frac{l}{2d_1}$ < than in

$a = 4$ for col
or knife

$a = 1$ for cc
ends no

The well-known formulas of Hodgkin-
n appear to be the first of substantial
ue, their constants being wholly em-
ical. They were applicable to only a
v forms of cross section, and entirely
dequate to the present engineering
actice.

Lewis Gordon determined the con-
nts for Tredgold's formula* from the

The formula of Tredgold is given in Tredgold on *Strength of Cast Iron and Other Metals*, 2d ed., page formula XIV.; and is

$$w = \frac{15300\, bd^3}{d^2 + 18\, l^2},$$

ich is readily written in the form

$$w = \frac{15300\, bd}{1 + 18\frac{l^2}{d^2}},$$

ere bd is the section of the column, =S, or K; and e length.

redgold gives credit for mathematical work to Dr. ing.

his is precisely the formula usually called Gor-'s, but it appears to be due to other authors, even he determination of the empirical constants for at st one case, probably cast iron, as appears from the mula as just quoted. But there can be no doubt of rdon having determined the constants for this mula for a variety of materials, though there may reasonable doubts of his being justly entitled to the dit usually accorded to him for the formula as a ole, and much more so for Rankine's modification t *where the radius* of gyration is introduced.

It came into quite
known as Gordon
has displaced the le
formula by the lea
thus producing the
mula in use to-day.

The form of this

$$w = \frac{f \mathrm{K}}{1 + a \frac{}{}}$$

where f and a are
the former being n
sistance to crushing
section and its least
spectively, at the n
length, l. The value

r f and a for any particular material, cept by actual column experiments—a ct which practically limits the formula a few materials, and even then to a omparatively few experiments. Hence, general formula whose constants con- ist of such factors as the coefficient of elasticity, modulus of crushing, &c., eems, by reason of the plentitude of these factors, very desirable.

These facts have caused search for such theoretical formula. One of the first formulas of this kind appears to have been proposed by Reuleaux, and published in *Der Constructeur*, the same being quoted by Ritter in his translation by Sankey of *Bridges and Roofs*, in 1879. See *Sankey's Ritter*, p. 345. The first edition was published in 1862.

This expression for rounded ends is of the form

$$w = \frac{cK}{1 + \frac{cKl^2}{8\varepsilon I}}$$

$$= \frac{cK}{1 + \frac{c}{8\varepsilon}\frac{l^2}{k^2}} \quad . \quad . \quad . \quad (121).$$

form of (121) is s
(120). That is t
given material is
times the section
constant times the
the length to the
cross section

Notwithstanding
form, (121) has
cability to all mate
are known, for
might not be kn
should give discre
empirical qualifyin
found which woul
the same for one n

An expression of
for round ends wa

in displacing 8 by π^2 in the denominator of (121). To account for this, Reuleaux treated the axis of the deflected column as a circle arc, while Crehore showed it to be a sinusoid. Hence Crehore's formula for rounded ends was

$$w = \frac{CK}{1 + \frac{Cl^2}{\pi^2 \varepsilon k^2}} \quad . \quad . \quad . \quad (122).$$

An expression identical with this for columns with round ends was given in an article by E. Hatzel, translated for VAN NOSTRAND'S ENG. MAG. for Sept., 1877; except in Hatzel's formula the moment of inertia was taken for an axis at other points in the section than the center of gravity; whence the radius of gyration k would be other than the principal radius.

Also Crehore showed that for flat ends the 2d term of the denominator of (122) should be divided by 4, and for one flat and one rounded end it should be divided by $\frac{1}{9}^6$.

Crehore's formulas were reproduced

article Professor Baldwin
dons these formulas, an
the discussion of the G
containing the least thickn
the Rankine formula conta
radius of gyration k. Ado
ter, its constants are deter
great number of experimen
for a variety of materials. B
was made to generalize b
and ε. The reason given i
constant in failing materi
point of rupture is unknow
be admitted, especially in
in compression the value

rtion of the cross section will be
l in incipient fracture. Existing
of experiment show that ε is
bly constant for compression of
wrought iron to near the failing
here it drops suddenly from 10
r cent. Hence if a tenth of the
is effected by 15 per cent. reduc-
ε, the resulting effect upon the
ection would be one and one-half
., a quantity which may be ig-
calculating the strength of col-
Hence we propose to compare
onal formulas with experiment
iving them up.

before taking up numerical facts
rve that the Reuleaux, Hatzel, or
formula differ from the rational
(97) for rounded ends, believed
ven for the first time in this arti-
ne form of the denominator. To
at this difference is an important
have from Fig. 21

$$ab : bB :: BA : \rho$$

$$ab : d_1 :: 1 : \rho$$

the last expression
first of (95).
whence

$$y_1 = \frac{t_1 l^2}{\varepsilon d_1 \pi^2}$$

in which t_1 is the com
is sufficient to cause
and a force which is d
Combining this with
have

$$t = \frac{T t_1 l^2}{\varepsilon I_1 \pi^2} + \frac{T}{K}$$

which, solved for T,
with rounded ends,

cept that t_1 is not the compressive resistance; Fig. 21 showing that it is only a part of it. In this expression t is the total compression represented by ad, and hence for rupture the total compression $t=C$ of Crehore's formula. But t_1, represented by ab, cannot be C, and hence some oversight appears to have been committed in making it C in formula (122). This explains the difference in form between (97) and (122); and gives sufficient cause for a disagreement of one or the other with experimental results.

In the developed denominator of (97) see (102) and (103), we have in the terms, $1+v$, (observing that $a=4$ for rounded ends) the identical denominator of (122). Hence to the remaining terms of the denominator of (103) we may look for the compensating quantity for the error $C=t_1$. The same fault is common to the Reuleaux, and to the Hatzel formulas though in the former a compensating effect appears to be provided in the values compounded of C and ε for use; and in

the latter in assuming the moment
ertia axis at or near the convex si
column.

Direct Comparison.—But the co
formula problem is one which cann
satisfactorily settled for practical
from a purely theoretical standpoi
therefore offer the following comp
of column formulas. As the num
may be considered as practically the
in all, the denominators only. are
pared.

Hence for the **denominator we**
for the

Gordon formula, $1+a\frac{l^2}{k^2}$

For round ends.	Round and fixed.	Fi
$a=\frac{4}{3000}$	$\frac{2}{3000}$	$\frac{}{3}$

kine formula, $1+b\frac{l^2}{k^2}$

	For round ends.	Round and fixed.	Fixed.
$b=$	$\frac{4}{36000}$	$\frac{16}{9}\ \frac{1}{36000}$	$\frac{1}{36000}$

By G. Bouscaren, C.E., and adopted by Keystone Bridge Company.

	For round ends.	Round and fixed.	Fixed.
$b=$	$\frac{2}{36000}$	$\frac{3}{2}\ \frac{1}{36000}$	$\frac{1}{36000}$

Reuleaux formula, $1+\frac{Cl^2}{8\varepsilon k^2}$ round ends

Crehore formula, $1+q\frac{Cl^2}{\pi^2\varepsilon k^2}$

	For round ends.	Round and fixed.	Fixed.
$q=$	1	$\frac{1}{2.28}$	$\frac{1}{4}$

New formula, $1+\frac{d_1^2}{2k^2}\left(\sqrt{1+\frac{\alpha t l^2}{\pi^2\varepsilon d_1^2}}-1\right)$

	For round ends.	Round and fixed.	Fixed.
$\alpha=$	4	$\frac{16}{9}$	1

From these expressions for the denominators, I have computed the results found in the following

LE OF VALUES OF DENOMINATORS FOR WROUGHT IRON.

	End.	Gordon.	Rankine.	Bouscaren-Keystone.	Crehore.	New.
cylinder.....	Rounded.....	5.8	7.4	4.2	9.28	5.30
ft.....	Round and flat.	3.4	3.84	3.4	4.66	3.74
=2 in.....	Flat.....	2.2	2.6	2.6	3.06	2.50
×12 in.....	Rounded.....	5.8	5.8	3 4	7.19	4.66
	Round and flat.	3.4	3.13	2.8	3.75	3.06
=10 ft.....	Flat.....	2.2	2.2	2.2	3.54	2.17
Hollow cylinder...	Rounded.....	4.00	1.26	1.13	1.38	1.30
Diam. 8 in. and 6 in.	Round and flat.	2.50	1.14	1.10	1.15	1.17
=10 ft..... ...	Flat.....	1.75	1.06	1.06	1.08	1.08
Hollow Rectangle..	Rounded	3.70	2.49	1.74	2 92	2.16
8×15 in. ×6×13 in,	Round and flat.	2.35	1.65	1.56	1.85	1.74
=30 ft.....	Flat.....	1.67	1.37	1.37	1.48	1.39
Beams.....		3.231	2.828	2.213	3.353	2.464

For the last two columns $t=C=40000$, $\varepsilon=?23370000$, and $\frac{t}{\varepsilon\pi^2}=\frac{1}{7000}$.

The first three formulas named in the table have been in quite general use, and yet the results shown in the table obtained from them differ in some cases nearly four hundred per cent., as for instance for the Gordon formula applied to hollow cylinders.

Results by the Crehore formula run high in value, one-half of all in the table going above those by the first three named formulas. It appears to give better results for hollow than for solid sections. But the large denominator gives small column load so that the formula errs on the safe side.

Results by the new formula are seen in every case but two, to fall between those of the first three "old reliable" formulas, and those two are outside only from one to three per cent. In comparing the means, we see that the new formula falls between the first three. If we reject the Gordon formula, which is now

going into disuse in the most
practice, and compare the new
with the 2d and 3d columns of re
find the mean by the new form
between the means of the 2d
columns. Indeed it differs but
two per cent. from the means o
columns. This appears the mo
able from the fact that the resu
2d and 3d columns are obtai
formulas which are probably th
use to-day in the engineering w

Comparisons have not been
other materials than wrought
the reason that wrought iron is
material in use at present in larg
where struts are required.

FORMULAS COMPARED WITH EXPE

To make the comparison of
formula with others in such a w
be yet more satisfactory to prac
the breaking load of 33 co
wrought iron, boxed and open
ed in bridges, has been
laced opposite the brea

by actual test in a testing machine.
n the table is placed the calculated
ng load as found from the Rankine
la, the Bouscaren-Keystone for-
and of the Rankine formula as
d over by Prof. Baldwin for new
ients, and given in his article above
ed to.
all cases except in Baldwin's for-
the coefficient in the numerator
ponding to the resistance to crush-
as taken at 40,000 lbs. In Bald-
formula all the coefficients were
as given in the Magazine. The
of ε for the new formula was used,
in a few cases, as given by the ex-
ents, when given. Corrections were
for pin friction, and for excessive
pacing for two cases. But these
ly effect the final result only 2 or
ent., because the number of cor-
is needed were so few. But the
nt. for the individual cases was
mes very considerable. The break-
ds stated are per square inch of
section of column.

writers to be from Love
I find no mention to
paper to the society.

The columns cited ar
forms of built columns

P. "Phoenix," hollow
flanged segment

K. "Keystone," octag
flanged segment
rically.

A. "American," two fl
to the flanges
beam.

S. "Square," hollow, t
channel bars riv

ed
or-
ge.
in
as-
It
on-
he
ve
00,
for
ors
ld-
re-
's.

I

p

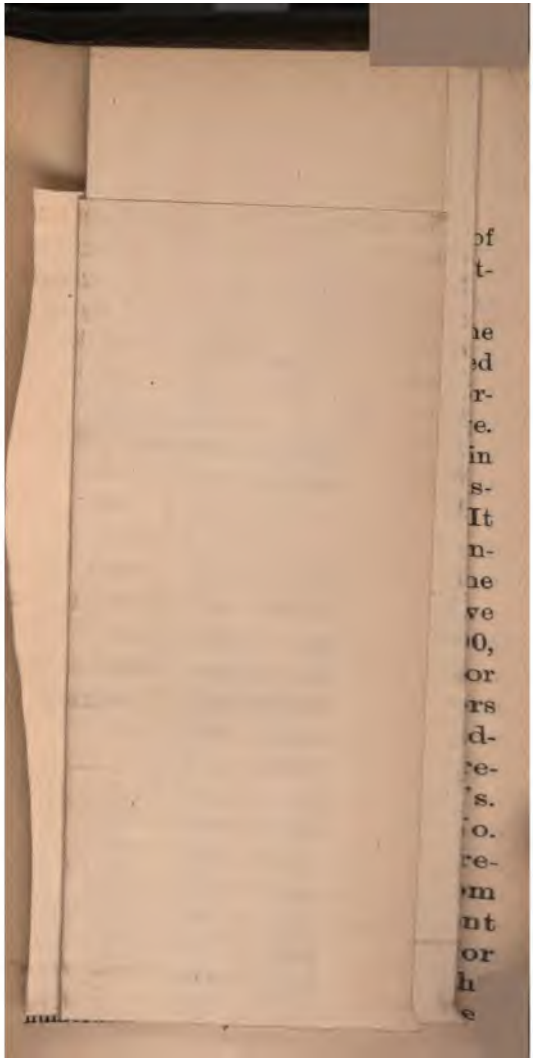
f

P

K

A

S.



worked out and given in the second set at the foot of Rankine's column of differences. The reduction from 40,000 to 36,000 is evidently too great as in one set the — values predominate, while in the other it is with the + values. An estimate made on this score would indicate that 38,000 would be about the best value for the Rankine formulas, the same indeed as is known to be in use by some Ry. Cos. It appears also that 40,000 is too great for the Keystone formula.

In glancing over the four last columns it is seen that in the first the — signs are chiefly among the results for flat ends, while the + signs predominate at the round ends. This indicates that the Rankine formulas are constitutionally at fault, or perhaps that the 1, $\frac{1}{9}^{6}$, and 4, coefficients to the last term of the denominator for flat, flat and round, and round, ends differ too much. As adopted in the Bouscaren-Keystone formula, viz.: 1, $\frac{3}{2}$ and 2, the discrepancies would probably be less, as indeed the column footings show. With these coefficients it would appear

that a better numerator would be about 38,000.

The column of differences due to the Baldwin formula are found to be far more satisfactory than either of the two preceding it. That is to say, the signs are more generally intermixed, the + and − values sum up more nearly equal to each other numerically, and the average discrepancy, without sign is smaller.

If, however, Nos. 20 and 34 were cut out from the table, the + and − sums would more nearly agree in value for the Rankine formula than for any of the others. But the mean error without sign would be higher than for any of the others, as indicated by the figures given Though the Baldwin formula still excels the Rankine and Keystone formulas in the smallness of mean error, yet the difference in + and − sums is greater.

But in every respect we find the last column of figures, which are due to the new formula, by far the most favorable. That is to say, the individual discrepancies in that column run lower in value

and differ less; and the m
discrepancy, is lowest. This, t
the fact that the theoretical v
and 4 are employed, where
formulas, they seem to differ
would indicate that the new
more nearly theoretically per
any of the others. But, in spit
advantages, the new formula is
nately, less convenient in applic
the others, and the question w
less arise in practice, whether
an important structure, to sa
additional hour required for
trustworthy results by the

the point of rupture, the formula may be used without hesitation. The point of rupture is here meant to be either for rupture by tension or compression. For such a material as cast iron it may be necessary to be guided partly by empiricism, especially for long columns, where the rupture may take place on the convex side by tension, because, in tension, the ε rapidly falls off in value, as rupture is approached. But in compression, ε is more nearly constant, and hence in short columns the formula will probably apply with greater exactitude.

The great difference in the ultimate resistances to tension and compression in cast iron will also cause an approach to rupture on the convex side, relatively, much sooner than for other materials, so that the rapidly diminishing value of ε will effect much shorter columns than otherwise. For cast iron in particular, therefore, it may be necessary to use a low value of the modulus of compression and of ε, and more so as the length increases, say 60,000 to 80,000 for the

which more satisfactory
tively new material, such
grades of steel, than an
cal formula which has no
that is, in cases where re
the compressive resistanc
known, and not of the c
empirical formulas.

MULTIPLYING EMPIRICAL

Where numerous results
exist for particular classes
may be possible to adopt
formula for each class, or ps
which shall give the resu
columns, or at least to repro
ticular values to which those
adapted, with

formulas, or require a modification of former ones.

In the above tables, comparisons were made only with formulas which are proposed to be perfectly general. But *Trautwine's Pocketbook* cites six different formulas by D. J. Whittemore, C. E., which, accordingly, would be required in computing strains for three of the forms of columns in the above table,and including only 17 of the experimental tests. Though Whittemore gives 12 different formulas, only 6 apply, and half of the experimental columns cited in the table are left by him without a formula.

In a recent paper by Prof. Wm. H. Burr, ten different formulas are produced from twenty-nine experiments. The experiments are mostly cited in the above table, and the formulas, when the whole ten are used, each for its intended form of column, give results agreeing remarkably well with the experiments. The mean error, regardless of sign, is 1720—a quantity less by about 33 per cent. than the mean by the new formula from

to tension is about 50,000 lbs. per square inch, and the $\frac{1}{5}$ part, or 10,000 lbs., is almost an absolute conventionalism among bridge builders, and may be regarded nearly in the sense of an absolute modulus of working resistance to tension.

Accordingly, $\frac{1}{5}$ is prefixed to column formulas (Rankine's and Gordon's) by many bridge engineers, as giving the proper fractional part of the ultimate resistance, which is to serve as the safe working resistance. But in empirical formulas, where the composition of the constants is unknown, this is a blind practice, because, from analogy with eye-bar strains, $\frac{1}{5}$ of the ultimate compressive resistance of the *material* is evidently meant, instead of $\frac{1}{5}$ of the resistance of the *column*.

This matter can be corrected in the Rational formulas, and it will be seen that the "$\frac{1}{5}$" coefficient to the empirical formulas for columns gives an unintentionally small safe load. A glance at formulas (114) and (120) will suffice for this. *Thus*, (114) shows that by using $\frac{1}{4}$

in (120), we only modify
by taking $\frac{1}{5}$ of the br
formula.

To indicate how the "
fected by the empirical a
the rational method, the
have been computed for
the columns in the princi

	No.	l	$\frac{l}{k}$	For breal
Flat end	6	15	5.2	
"	23	24	7.1	
"	29	28	9.3	
Rounded	16	20	8.5	
"	36	34	12.	

breaking load, does not strain the material to more than about one-seventh of its ultimate resistance. But fortunately this difference is on the safe side, so that the majority of existing columns in bridges are amply strong.

The figures in the last column are all less than 8000 lbs., the maximum strain assumed as allowable, that is, the fifth part of 40,000 lbs. The differences are to be considered as provisional, for preventing the strains rising in excess of the 8000 lbs. by buckling or otherwise.

By taking $\frac{1}{4}t$ in place of $\frac{1}{5}t$ for computing the last column of figures, perhaps as close an agreement is obtained with the column preceding it as is possible with a simple fraction. From this fact it appears that the "factor of safety" usually employed for columns has been in the neighborhood of $\frac{1}{4}$, or that the working resistance of iron in columns has been from 5000 to 6000 lbs. per square inch, compression. But as a high value of the working resistance to com-

$$t=\frac{\mathrm{T}hd_1}{\mathrm{I}\cos nl}\pm\frac{\mathrm{T}}{\mathrm{K}} \quad . \quad . \quad . \quad (124).$$

$$\left.\begin{array}{l}\text{where } n^2=\dfrac{\mathrm{T}}{\varepsilon\mathrm{I}}=\dfrac{t'\mathrm{K}}{\varepsilon\mathrm{I}}=\dfrac{t}{\varepsilon k^2}\\ \text{also } \dfrac{\mathrm{T}}{\mathrm{K}}=t', \text{and } \dfrac{\mathrm{T}}{\mathrm{I}}=\dfrac{t'}{k^2}\end{array}\right\} \quad . \quad (124a).$$

These quantities may, indeed, be introduced with probable greater convenience for application, giving

$$\frac{t}{t'}=\frac{hd_1}{k^2\cos\frac{l}{k}\sqrt{\frac{t'}{\varepsilon}}}\pm 1 \quad . \quad . \quad (125).$$

the + sign being used to obtain the max. compressive strain, and the − sign for the max. tensile strain. The piece will fail from the first or second, as depending on the nearest approach of that strain to the modulus of crushing or of rupture. For wrought iron the + sign should be used, while for cast iron, except when h and l are very short, the − sign. The max. strain per square inch is t, tension or compression as just indicated, while t' is the actual average compressive force

which is applied to the semi-column per square inch of the cross section. For instance, if the section be 10 square inches and the compressive force applied be 100,000 lbs., $t=$ 10,000 lbs. But the max. t' resulting strain in the semi-column may be $t=20{,}000$ lbs. per square inch. Also h is the distance from the center of gravity of the end cross section to where the compressive load is applied; d_1 the distance from the center of gravity of the middle section to the side, concave or convex, where the strain t is reckoned; k the radius of gyration of cross section; l the length of semi column, and ε the coefficient of elasticity.

When d_1 is taken on the convex side, and the 1st term 2d member $= 1$, we will have tension $t=o$.

Though a direct determination of t is effected for an assumed value of t', yet the converse will doubtless most frequently be desired, but cannot be obtained by a direct simple solution. A few trial values of t' however will give a

tly close value for t.

For a case like Fig. 29 it is only necessary to put $\frac{l}{2}$ instead of l in (125); whence, for AB=l, AC=h, and for pin bearings at C and D,

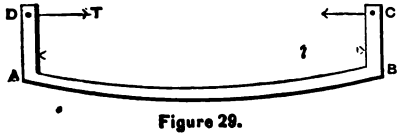


Figure 29.

$$\frac{t}{t'}=\frac{hd_1}{k^2\cos\frac{l}{2k^2}\sqrt{\frac{t'}{\varepsilon}}}\pm 1 \quad . . \quad (126).$$

If there be a load, P, at the middle of the semi-column, acting transversely and downward in Fig. 29, a suitable expression is obtained by substituting M from (23) in (8), observing that for (23), l and P should be changed to $\frac{l}{2}$ and $\frac{P}{2}$ and that the essential sign of M is $-$.

Hence, for pin bearings, and a load P at the middle

$$= \frac{d_1}{I}\left(\frac{P}{2n}\tan g\ n\frac{l}{2} \quad \frac{Th}{\cos n\frac{l}{2}}\right) \pm \frac{T}{K} \quad . \quad (127).$$

n which I = the principal moment of inertia of the cross section K, T the whole compressive force applied, and n as given above. P is — for acting upward.

In order that the flexural moment at the middle of the semi-column be zero, the parenthesis should be zero. Whence

$$h = -\frac{P}{2nT}\sin n\frac{l}{2} \quad . \quad . \quad . \quad (128).$$

This value of h is expected to be negative, and it implies that when P acts downward in Fig. 29, the arms h=BD= AC must be laid off downward.

This equation is applicable to upper chords of deck bridges which carry a floorbeam at the middle of each panel, and for determining the position of the pin bearing such that the max. load shall only cause compressive strain in the middle panel length. The chord be considerably heavier termined.

By making $h=o$ in (127), we obtain e expression for a column with pin arings against the middle of which ere exists the lateral thrust P, viz.:

$$=\frac{Pd_1}{2nI}\text{tang}\,d\frac{l}{2}\pm\frac{T}{K} \quad . \quad . \quad . \quad (129).$$

In deck bridges where numerous floor ams rest on each panel length of the per chord, as is sometimes the case th wooden floor beams on iron truss- we have each panel length serving e a column under cross loading, the ter being nearly uniformly distributed. An expression for such a case is ob- ned from (35) and (8). Hence for n bearings

$$\frac{wd_1}{n^2I}\text{tang}\,n\frac{l}{4}\text{tang}\,n\frac{l}{2}\pm\frac{T}{K} \quad . \quad . \quad (130).$$

$$=\frac{wd_1}{n^2I}\left(\sec\,n\frac{l}{2}-1\right)\pm\frac{T}{K}$$

ere w is the loading per unit length.

With Fixed Ends and Compression. Where the semi-column is fixed at the ds, as in continuous upper chords,

h (8) for a load P at
ce

$$\frac{l}{4} \pm \frac{T}{K} \quad . \quad . \quad . \quad (131).$$

ain at either the mid-
anel length.
distributed load, w,
for continuous upper
(78) and (8)

$$\pm \frac{T}{K} \quad . \quad . \quad . \quad (132).$$

panel length, and
greatest.
ill have from $x = \frac{l}{2}$

$$\frac{T}{K} \quad . \quad . \quad . \quad (133).$$

distributed loads
strain due to the
nselves, as in up-

per chords of through bridges; and the inclined end posts. In the latter however gravity acts with a transverse component only, which is considerably less than where the piece is horizontal.

Other cases, such as for one pin bearing and one fixed end, is provided for in (63), (64), (65) and (8), two floor beams and pin bearings in (49), &c., &c.

Pin Bearings and Tension.—In lower

Figure 30.

where the longitudinal force is tension, chords of bridges and other cases

we may combine (20) with (8), and obtain an expression adapted to Fig. 30.

$$= \frac{T}{K} \quad . \quad . \quad . \quad . \quad (134),$$

where $l = AS$ and $i = BC$. Here d_1 will usually be measured on the convex side of beam, and where the + sign will generally be required. Also t will probably always be constant.

The formula for Fig. 31. will be ob-

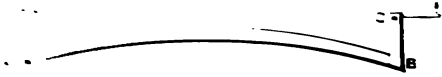


Figure 31.

tained by replacing $\frac{i}{2}$ in 134 by $\frac{i}{2}$, giving the two bearings at C and D.

$$= \frac{T}{K} \quad . \quad . \quad . \quad 135.$$

$i = AB$, and $i = AD. = BC$.

load is applied at the middle,

acting downward, we have, by aid of (19) and (8), observing that P in (19) is the half of P here,

$$f = \frac{Pd_1}{2nI}\frac{\left(e^{n\frac{l}{2}} - e^{-n\frac{l}{2}}\right) - 2Th}{e^{n\frac{l}{2}} + e^{-n\frac{l}{2}}} \pm \frac{T}{K} \quad . . \quad (136).$$

In order that the flexural moment at the middle of the beam shall be zero, the last term, 2d member, should be zero, which condition determines h.

Also h may be zero for a link or eye bar.

For a uniformly distributed load of w per unit length, we have for pin bearings

$$f = \frac{wd_1}{n^2 I}\left\{1 - \frac{2}{e^{n\frac{l}{2}} + e^{-n\frac{l}{2}}}\right\} \pm \frac{T}{K} \quad . . \quad (137).$$

Fixed Ends and Tension.—For fixed ends we may combine (72) and (8) for a load at the middle, and for a distributed load (73); and these give strains at the end of the panel length. For the middle employ (74).

inertia as dotted, are exclusive of lat-
ing,

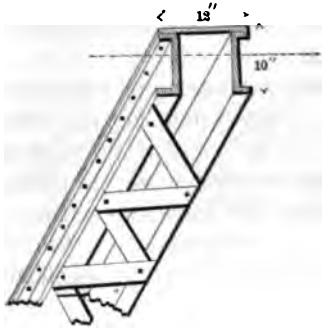


Figure 32.

K = section = 12.875 sq. in.
I = 205, axis 3" from plate.
k^2 = 15.93 in. = radius of gyration.
d_1 = 3.36 in. plate side.
" = 7. " opposite side.
w = wt. per inch length 3.6 lbs.

As a vertical column with knife bear-
gs or pins without friction at center of
avity of sec. and with a max. strain
10,000 lbs. per square inch, the resist-

proportional to w when T is constant, so that if this load be increased to 40 lbs. per inch or 480 lbs. per ft., the strain will be increased to 5,559 lbs. per sq. in., while by the ordinary formula, ignoring T, it is 4,894 lbs. As an end post inclined at 60° with the horizon the strains will be reduced by a half.

Next, let the loading be P at the middle, of 1,000 lbs., with T and other data as before. As a column in pin bearings, Eq. (127), with $h=o$, gives for the strain due to P 1,090. lbs. per sq. in. By the ordinary formula it is 985.

Transferring the pins to the top of end sections, other conditions as in the last, then $h=3.36$ in. in (127), and the strain due to 1st term of 2d member of (127) is $1090+578=1668$.

To find where to place the pins so that the moment of strain and the middle shall be zero, we have from (128) $h=-6.34$ inches, which is within $\frac{2}{3}$ of an inch of the bottom.

As a last example, let the aggregate section of an iron lower chord be 4″

B'D. OCT 1913